NRL Report 6814

A Lunar Radar Navigation Concept

[Unclassified Title]

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May 5, 1969

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ABSTRACT

A navigation system concept is described that utilizes the moon as a reflector in a bistatic radar system. By measuring the range and range rate of the receiver relative to the moon, the location of the receiver on the earth's surface can be determined in both longitude and latitude. Lunar radar observations have shown that lunar range and range rate measurement accuracies equivalent to ±30 m can be achieved. By placing a transponder on the moon, the basic measurement accuracy could be improved by a factor of 5. However, for operational measurements where rapid readout is required, these accuracies would probably be degraded by a factor of 5 to 10. Placing three transmitters at appropriate locations on the earth's surface will provide worldwide coverage. A transmitter with an average power of 2 MW and a transmitting aperture of 170 m would supply a S/N ratio sufficient for reliable position determination with a dipole antenna receiver. One possible radar receiver configuration incorporating both a search and track mode is given.

A mathematical analysis of the coverage and the effective position accuracy indicates that (a) worldwide coverage is available, (b) time coverage is restricted to 50 percent on the average, but the time distribution of the coverage varies over a monthly period, (c) at low latitudes, the effective location accuracy is a function of the moon's declination, and (d) the optimum accuracy is obtained at high latitudes.

While restricted to some extent in coverage as described above, and more sensitive to systematic errors in the ship's velocity, the lunar radar navigation system, when compared with other worldwide radio navigation systems such as Omega and Transit, can achieve higher accuracies, is less vulnerable to jamming, and can also provide an independent, one-way communication channel. With improved technological developments and active reflectors on the moon, position accuracies of the order of several meters appear possible.

PROBLEM STATUS

This is a final report on one phase of the problem, work is continuing on the project.

AUTHORIZATION
NRL Problem A01-35
Project 33404, Task 4896

Manuscript submitted October 4, 1968.

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INTRODUCTION

Many different methods have been employed to determine the position of ships on the ocean or of other unknown locations on the earth's surface. To achieve the requirements of increasing accuracy, and all-weather, global, and 24-hour coverage, several radio navigation methods have been developed. Among these, the Transit satellites and the Omega system have been the most promising under development. However, practical considerations have necessitated some compromise between the various requirements. In particular, propagation uncertainties in Omega and orbit uncertainties of the Transit satellites limit the potential accuracy that can be achieved.

The lunar navigation system to be described in this report can overcome some of the limitations of the other systems and should improve the potential accuracy by an order of magnitude. This is achieved primarily by measuring the position of the moon simultaneously with the position of the unknown location. To obtain optimum accuracies (±5 m), a lunar transponder may be required. In addition, the system is less vulnerable to jamming and provides one-way communication between the master station and the unknown location. The basic disadvantage of the system is that the moon is above the horizon for only about 50 percent of the time.

BASIC CONCEPT OF THE LUNAR NAVIGATION SYSTEM

The lunar navigation system utilizes the moon as the reflector in a bistatic radar system where the transmitter is at a known location and the receiver determines its position on the earth's surface from measurements of its distance and velocity relative to the moon. The transmitting station measures simultaneously its own distance and velocity relative to the moon and corrects the transmitted signal for variations of the lunar topography and the lunar ephemeris errors. The receiving station derives its position by applying the known lunar ephemeris to the range and velocity measurements. The two measurements provide the necessary corrections to both longitude and latitude of the unknown location. To provide worldwide coverage with some overlap, three ground-based transmitting stations are required, separated by 120° in longitude and placed at moderate latitudes. The system is worldwide except for the limitations imposed by the time-varying geometrical configuration of the earth-moon system, which is discussed in the next two sections.

BASIC LIMITATIONS OF COVERAGE

There is a basic limitation of coverage due to the availability of the moon at a given time for a given location. The moon is available at any given location only when it is above the horizon. The altitude of the moon (h) at a given location is given by

$$\sin h = \sin \varphi \sin \delta + \cos \varphi \cos \delta \cos \gamma \ , \tag{1}$$

where

φ is the geocentric latitude of the receiver,

δ is the declination of the moon, and

γ is the local hour angle.

The limits of visibility of the moon are given by Eq. (1) for $h = 0$. This yields the usable local hour angle for a given latitude and lunar declination as expressed in Eq. (2):

$$\cos \gamma = -\tan \varphi \tan \delta \ . \tag{2}$$

In Fig. 1, the usable hour angle per day is plotted as a function of latitude and different declinations of the moon. For a declination of 0°, the limiting hour angle is ±90° or 12 hours for all latitudes. For positive declination, the available observation time per day increases with latitude until 24-hour coverage is obtained at high latitudes. At large negative latitudes for the same day the available observation time is reduced. The opposite behavior as a function of latitude occurs for negative declination. As the moon changes its declination from -28° to +28° in two weeks, the moon at latitudes close to the poles will be observable for at least two out of every four weeks.

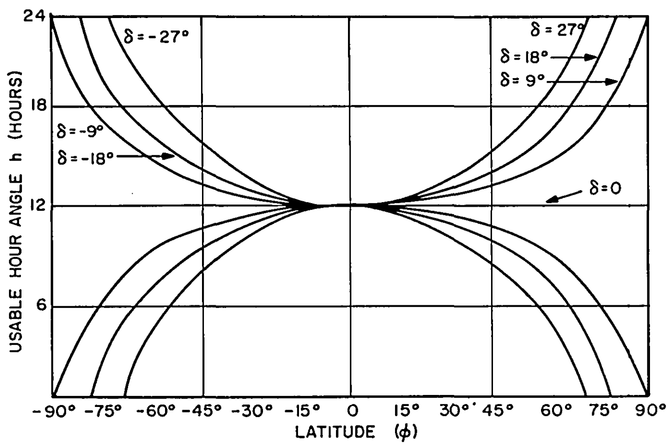

Fig. 1 - Visibility of the moon vs latitude and the moon's declination

ERROR ANALYSIS

The locus of all points on the earth's surface which are equidistant from the moon at a given time is a circle on the earth's globe. This circle is located in a plane normal to the sublunar distance with the center of the circle positioned below the sublunar point. (See Fig. 2.) As the earth rotates around its axis, each point on the earth's surface will either approach or recede from the moon, except for the stationary points on the great circle which is located in a plane passing through the sublunar point and the earth's axis of rotation. The loci of equivelocity points are small circles on the globe, parallel to the plane containing the great circle of zero velocity. The range and range rate measured

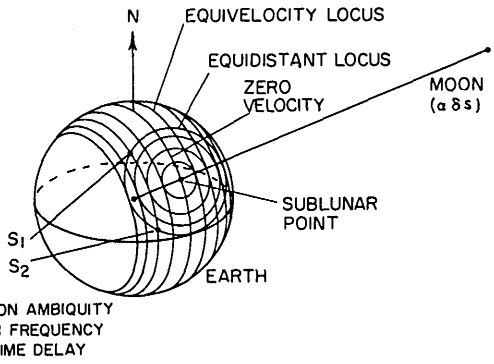

Fig. 2 - Loci of lunar equidistant and equivelocity points

by the observer will establish an equidistance and an equivelocity circle. The intersection of these two circles determines the location of the observer.

In general the intersection of the equidistant and equivelocity circles corresponding to one measurement will provide two possible locations on the earth's surface (Fig. 2). However, this ambiguity can be removed, since the approximate location of the measured position is generally known. The angle of intersection of the two circles determines the effective accuracy with which the two required quantities (longitude and latitude) can be determined. In general, the accuracy of the longitude is less dependent than that of the latitude on the angle of intersection of the two circles. The latitude error increases as the two circles approach each other tangentially. Since the sublunar point on the earth's surface changes with both a daily and a monthly period, the relative position of the two circles will change with time, and thus the effective error of the longitude and the latitude will be a function of both time and position.

In Appendix A, the effective error for both longitude and latitude has been derived. In the computations it has been assumed that the measurement errors of time delay and frequency are equal for the most efficient operation, and that the corrections for errors in the lunar ephemeris and for variations of the lunar topography have been made by the transmitting station. The effective errors are given by Eqs. (3) and (4):

$$\sigma_{\Delta x} = \frac{\sqrt{\sin^2\varphi + \cos^2\varphi \tan^2\delta - \sin 2\varphi \tan\delta \cos\gamma}}{\sin\varphi - \cos\varphi \tan\delta \cos\gamma}\,\sigma_m , \tag{3}$$

$$\sigma_{\Delta y} = \frac{1}{\sin\varphi - \cos\varphi \tan\delta \cos\gamma}\,\sigma_m , \tag{4}$$

where $\sigma_{\Delta x}$, $\sigma_{\Delta y}$, and σ_m are the standard deviations in linear dimensions (meters) of longitude, latitude, and of the measurements, respectively.

These expressions give the effective errors of Δx and Δy as a function of latitude φ, the local hour angle γ, and the moon's declination δ. For some limiting cases these errors can be evaluated in a simple manner. For example, for $\delta = 0$ and $\sigma_m = 1\ m$,

$$\sigma_{\Delta x} = 1\ m , \quad \sigma_{\Delta y} = \frac{1}{\sin\varphi}\ m .$$

The effective error of the longitude is constant and equals the measurement accuracy at all locations. However, the error in the latitude increases as the latitude approaches zero (equator). Similarly, if $\varphi = 0$ and $\delta \neq 0$,

$$\sigma_{\Delta x} = \frac{1}{\cos \gamma}\ m\ , \qquad \sigma_{\Delta y} = \frac{1}{\tan \delta \cos \gamma}\ m\ .$$

In this case, the errors of both longitude and latitude are a function of the local hour angle and are minimum at transit.

Data for a complete coverage diagram based on Eqs. (3) and (4) and the visibility of the moon discussed previously are derived in Appendix A and given in Tables A1 through A7. In general, the effective accuracies are best at high latitudes.

MEASUREMENT ERRORS

The effects of the ionosphere and the atmosphere on the time delay and frequency are negligible at microwave frequencies for measurement errors of the order of 100 m. For more accurate measurements, corrections for typical atmospheric conditions can reduce errors by an order of magnitude. This may become necessary if the full measurement accuracy of the lunar transponder is to be utilized.

The main modification of the radar signal is caused by the complex reflection characteristics of the moon. The varying topography will spread the leading edge by extending the buildup of the pulse anywhere between 5 and 20 microseconds (Fig. 3 shows a typical radar echo). In addition, the frequency is spread by the apparent rotation of the moon by about one part in 10^8. However, by measuring the frequency spread in the first intervals of range (about 10 to 100 μsec), the spread is reduced by an order of magnitude. The measurement consists of finding and establishing the centroid of both the leading edge and the frequency spectrum with an accuracy corresponding to 1 μsec in time and a frequency shift of one part in 10^{10}. This is probably best accomplished by utilizing a tracking gate in range and a direct frequency count to establish the mean time delay and frequency of the received signal.

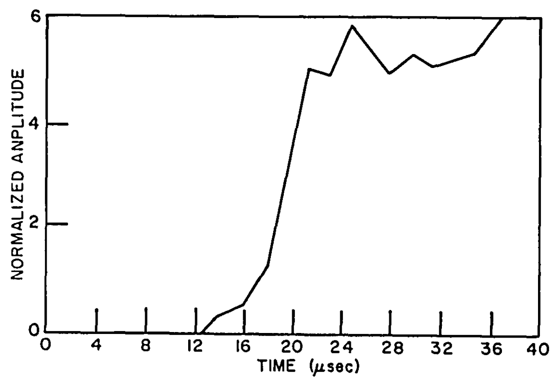

Fig. 3 - Lunar radar echo of May 15, 1968 (Observation 187, No. 75), integration time 0.1 sec

Measurements of the lunar distance with the NRL lunar radar system, which employs a basic 1.2-μsec range resolution, have shown that the internal consistency can be maintained to within ±0.1 μsec or ±30 m in flat areas. The range variations for a few adjacent areas are given in Table 1. However, these measurements are obtained from the first point of reflection of the echo. Measuring to the mean height of a lunar area should improve the consistency of the range measurements and reduce the frequency of lunar range corrections imposed on the transmitter.

Table 1
Range Variations of Several Locations on the Moon

Time Interval	L	B	Δr (m)	General Area on Moon
1/2 hour	-6.95 -7.11	-1.72 -1.72	40	Mosting
1/2 hour	-7.57 -7.63	-3.08 -3.04	-30	Mosting
1/2 hour	6.85 6.60	5.73 5.91	-10	Hyginus
1 day	1.43 1.44	6.78 6.69	-20	Ukert
1 month	-3.79 -3.78	1.90 1.71	0	Sinus Medii

Frequency measurements have been achieved by MIT (1) with accuracies of one part in 10^{11} on the moon, while JPL (2) has obtained accuracies of the order of one part in 10^{12} in measurements of frequency shift with transponders from the Lunar Orbiter series. The frequency spectrum of a lunar echo, as derived from an NRL cw lunar radar measurement, is shown in Fig. 4. If the spectrum from only the initial return is obtained, the deviations from a gaussian spectrum and the width will be reduced, thus improving the accuracy with which the centroid of the spectrum can be obtained.

EFFECT OF SHIP'S VELOCITY ERROR ON POSITION LOCATION

Estimates of the ship's velocity may contain either random errors that vary with time or systematic errors that remain constant for several hours. Random errors with short periods (less than the observational interval) will spread the frequency spectrum and, for random velocities of less than 1 km/hr, will not materially affect the accuracy of the position determination. The effect of slowly drifting random errors of the ship's velocity can be reduced by obtaining several measurements in order to average the effective position error.

The systematic errors in the ship's velocity that can be tolerated without degrading the position accuracy are proportional to the effective error, as discussed in the error analysis section.

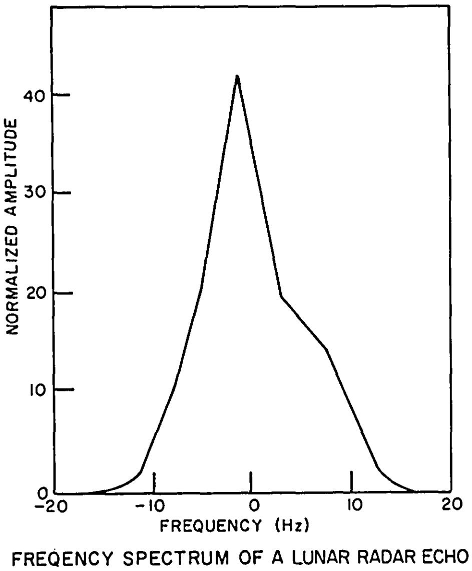

Fig. 4 - Frequency spectrum of a lunar radar echo

The effect of the ship's velocity error on the position accuracy has been derived in Appendix B as a function of the ratio of the velocity error to the measurement error. The results are given by

$$R_1 = \frac{\sigma_{\Delta x_v}}{\sigma_{\Delta x_m}} = \frac{a_{12}(\sin^2\gamma + a_{12}^2)^{1/2}}{(a_{22}^2 + a_{12}^2)^{1/2}} \frac{\sigma_{\Delta v/\omega_e}}{\sigma_m} \tag{5}$$

and

$$R_2 = \frac{\sigma_{\Delta y_v}}{\sigma_{\Delta y_m}} = \sin\gamma\,(\sin^2\gamma + a_{12}^2)^{1/2} \frac{\sigma_{\Delta v/\omega_e}}{\sigma_m}\,, \tag{6}$$

where

γ = the local hour angle,

$a_{12} = \cos\gamma \sin\varphi - \cos\varphi \tan\delta$,

$a_{22} = -\sin\gamma \sin\varphi$,

$\sigma_{\Delta x_v}$, $\sigma_{\Delta y_v}$ = the errors caused by the velocity error on the longitude and latitude determination,

$\sigma_{\Delta x_m}$, $\sigma_{\Delta y_m}$ = the errors caused by the measurement errors on the longitude and latitude determination, and

$\sigma_{\Delta v/\omega_e}$, σ_m = the velocity and measurement errors respectively.

Tables B1 through B7 represent R_1 and R_2 as a function of the local hour angle γ, the geocentric latitude φ, and the moon's declination δ for the case when $(\sigma_{\Delta v}/\omega_e)/\sigma_m = 1$. The total effective errors $\sigma_{\Delta x_T}$ and $\sigma_{\Delta y_T}$ of longitude and latitude are given by

$$\sigma_{\Delta x_T} = \sqrt{1 + R_1^2}$$

and

$$\sigma_{\Delta y_T} = \sqrt{1 + R_2^2} \ .$$

For velocity errors comparable to the measurement errors, the effective position error increases on the average by about 20 percent. For the assumed measurement error of ±300 m, this corresponds to a velocity error of ±3 cm/sec. If the velocity error exceeds the measurement error by a factor of four, then the total error is increased by 100 percent on the average. For larger velocity errors, observations over several hours are required to determine both the position and velocity corrections, if the full measurement accuracy is desired.

DESIGN CONSIDERATIONS FOR LUNAR RADAR NAVIGATION SYSTEM

In considering the system parameters for a lunar bistatic radar system, it has been assumed that the operational measurement accuracy will be degraded by an order of magnitude from that obtained from lunar radar astronomy measurements (1-3). Furthermore for optimum efficiency, the time (range) and frequency measurement errors are made equal as discussed in the section on measurement errors. This corresponds to a 1-μsec time resolution and a frequency resolution of one part in 10^{10}, which are equivalent to a position error of ±300 m.

Another consideration concerns the radar frequency to be employed with the lunar radar navigation system. Since one of the basic measurements is the doppler shift due to the relative motion of the ship's location with respect to the moon, the higher the radar frequency the greater the doppler shift and in turn the faster a readout can be obtained for a given accuracy. On the other hand, the complexity of a transmitting system that supplies sufficient energy to a dipole receiving system increases with increasing radar frequency. These considerations will require a compromise between the complexity of the radar transmitting system and the tolerable observation time. With larger receiving antennas, higher frequencies, or more complex transmitters, a higher accuracy and a more rapid readout can be obtained. An indication of the relation between these factors and frequency is given by

$$f_c \propto \frac{1}{\epsilon_{\Delta f}\ T_0\ A_r\ G_t\ P_t} \ ,$$

where

$\epsilon_{\Delta f}$ = the measurement accuracy in frequency,

T_0 = the observation time,

A_r = the receiving aperture,

G_t = the gain of the transmitting antenna, and

P_t = the transmitter power, and 500 MHz $< f <$ 10,000 MHz for useful operation.

There are several basic requirements that the bistatic radar system has to satisfy:

1. A S/N ratio of 20 dB minimum is assumed at the receiver. If the receiver is to be made as simple as possible and to require no pointing by using a simple dipole as the receiving aperture, the major effort then has to be applied at the transmitter to provide sufficient energy at the receiving terminal. To obtain an indication of the type of transmitter that may be required, the parameters used in Table 2 are applied in the radar equation to supply the needed S/N ratio. If we assume a 1000°K receiver, the effective S/N is

$$S/N = \frac{P_t \; A_t \; \sigma \; G_r}{(4\pi)^2 \; R^4 \; KTB} \approx 100 \;,$$

where

σ = the radar cross section of the moon,

G_r = the gain of a dipole,

R = the distance to the moon,

K = Boltzmann's constant,

$\sigma = 4 \times 10^{-4} \times \pi b_0^2$,

b_0 = the mean lunar radius, and

B = 5 Hz.

Table 2
Transmitter Characteristics for a Dipole Receiver

Characteristics	Quantity
Average Power Transmitted, P_t	2 MW
Antenna Diameter	170 m
Pulse-Compression Ratio	2×10^5
Frequency Stability	1 part in 10^{12}
Range Accuracy	1 μsec

If smaller transmitters are desired, some tradeoff between the receiver and transmitter can be made. For example, if instead of a dipole a 1-m antenna is used with the receiver and the wavelength is changed to 3 cm, the transmitting antenna need only be

30 m with an average power of 60 kW. This will also reduce the required readout time by a factor of 10.

2. The time at the transmitter and the receiver has to be synchronized within the basic measurement accuracy. Thus, for example, for an accuracy of 300 m, the two clocks have to be known within 1 μsec. The use of atomic clocks is clearly indicated, and periodic resetting of the receiver clock is required. For a clock with a frequency stability of one part in 10^{13} (thallium), the receiver clock would have to be reset every two months.

3. Special codes may be used to reduce the time of search in the uncertainty interval of the range measurements. In general, a search should be completed within a few minutes.

4. Special-purpose computers have to be available at both the transmitting station and the receiver to convert the known information of the lunar ephemeris and the position of the transmitter to range and range rate as a function of time. The expected receiver position is introduced to determine an uncertainty interval from which the corrected position can then be obtained.

Operationally, it is expected that the transmitter will continuously illuminate the moon, but that the coding interval and its start will be exactly controlled by the atomic clock of the transmitter. The receiver will then tune in on the received signal and search for its position. Once the position is found, the receiver will continue to track the position and indicate the motion of the ship or vehicle in latitude and longitude. The effective error of the indicated position can be simultaneously displayed by incorporating in the computer the information presented in the tables of Appendix A. Details of one possible receiver configuration are given in the next section.

RECEIVING SYSTEM

The ultimate goal in designing and developing the receiving system is a package that provides automatic latitude and longitude information with little or no human adjustment. This concept is possible because all the tradeoffs for receiver simplicity will be vested in the transmitter. Since simplicity is of paramount importance, the design goal for the receiving antenna is a simple dipole or whip antenna. This eliminates the need for knowing where to point the antenna. The rest of the receiving system will utilize integrated circuits, microminaturization, and modular construction. It is envisioned that the final packaging of the receiver will be of the same volume and size as today's shipboard Loran C navigation receivers.

Figure 5 shows a proposed block diagram of the receiving system. Since biphase pulse compression is assumed in the transmitted signal, the receiver must decode the transmission in order to extract the necessary information. Also, the receiver must pass the signal through fairly narrow filters, which requires that the local oscillator (LO) chain have coarse frequency tracking to correct for the earth's rotation. This is achieved by squaring the i-f signal to remove the biphase code and then, with a frequency-following filter and appropriate counters, developing the control signals for adjusting the LO so that the signal will pass through the narrow filters used.

Once the LO frequency is correctly set, the signal is fed to ten correlators and filters (the number required is determined by operational requirements) to determine the

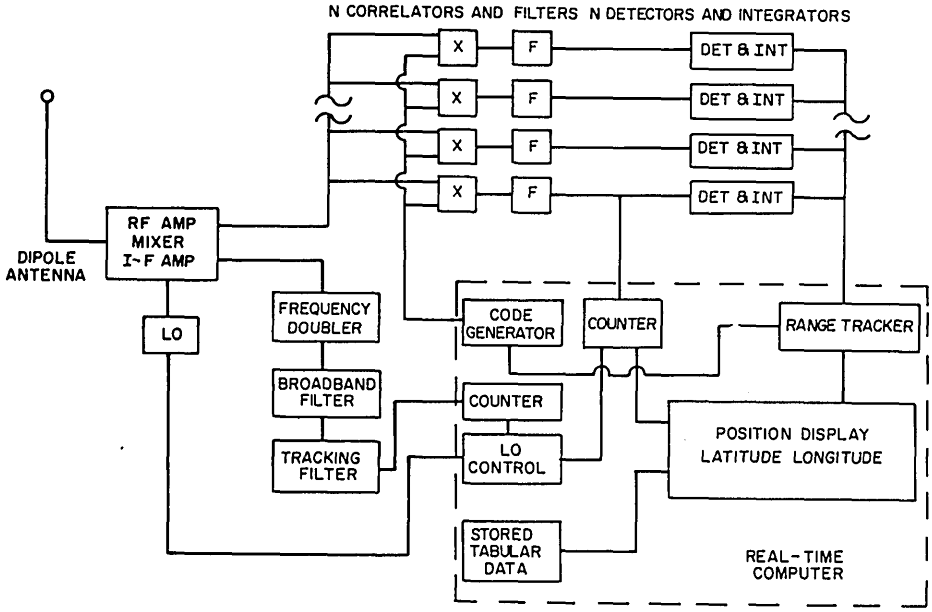

Fig. 5 - Navigation receiver

range or time of flight of the signal. The code generator will develop the proper codes which will be correlated with the incoming signal to provide coarse range information. The range tracker must sense a signal; otherwise, it will command the code generator to increment the code until a signal is found. Once the signal is found, the range tracker switches into the fine range mode to measure the range or time specified. Also, fine frequency will be determined after the fine range is established, and this is accomplished by using the output of one or more range channels and feeding its signal to an axis-crossing counter to determine precisely the frequency of the received signal. The range and frequency will be fed into a computer, which will display latitude and longitude automatically. The acquisition time is determined by the required range and frequency resolutions and the initial uncertainty of the position under observation. Assuming 1-μsec range precision and 100-km position uncertainty, the acquisition time is

	Maximum (sec)	Average (sec)
Lock in coarse frequency	40	20
Lock in coarse range	60	30
Lock in fine range	100	50
Measure frequency	100	100
Total time for acquisition	300	200

Once the signal has been acquired, updating in range is fairly quick (10 sec). However, to determine the frequency will still require 100 sec. The times used are for a particular set of parameters. Tradeoffs in performance and system complexity will alter the acquisition and updating times required.

DISCUSSION AND CONCLUSIONS

To evaluate the performance of the lunar radar navigation system, it may help to compare it with the Transit system, which is similar in fundamental concept. The differences of the lunar navigation system with respect to Transit can be summarized as follows:

1. A natural satellite is employed as the reference point in the sky. This obviates the necessity of periodically injecting artificial satellites for this purpose, but restricts to some extent the coverage, as a function of time and position, which would be available with a sufficient number of artificial satellites.

2. The orbit of the moon is well known, and any minor corrections to its orbit can be made by the transmitter, which continuously monitors the motion of the moon. This makes it possible to increase the effective accuracy of the range and range rate data as compared with Transit, where the orbit of the satellite has to be updated and then predicted for a period of several hours.

3. The relatively slow rotation of the earth as compared to the rapid motion of the satellite requires a higher frequency stability at the receiver for a comparable accuracy in position measurements. This fact also makes the position determination more sensitive to errors in the ship's velocity.

4. One-way communication is available with some minor modification.

5. The moon is not vulnerable to hostile interference and is much more difficult to jam.

6. Finally, due to the large distance of the moon from the earth, the system may be considered in the future as a navigational aid to lunar and close interplanetary spacecrafts.

At present, the achievable accuracy is limited technologically by the accuracy of the available atomic clocks and the need to reset them periodically. With the future development of atomic clocks and the placement of transponders on the moon, accuracies of several meters should be possible. On the other hand, for stationary locations where observations can be made for several hours, atomic clocks can be reset automatically and simultaneously with the position measurements.

To evaluate more closely this concept, it will be necessary to utilize one of the many radar transmitters now available for testing the search and track mode of a receiver in an assumed unknown location and to establish more definite performance criteria. If this concept should prove itself practical, it may then be feasible to consider placing a transponder or a corner reflector on the moon to increase the accuracy as well as reduce the time of observation.

Future improvements of the lunar radar navigation concept may include the utilization of additional radio reference points in the sky. The additional radio reference points can be made available by placing isotropic reflectors at the libration points of the moon.

The applicable libration points are located in the moon's orbital plane and consist of the two stable equilibrium points L_4 and L_5 (4) (equilateral points) at ±60° in right ascension relative to the moon, and one semistable equilibrium point L_3 (collinear point) opposite the moon. If a reflector is injected at any one of these points, it is expected to stay indefinitely in the general vicinity of the particular equilibrium point. The small periodic motions of the reflectors around the equilibrium points can be compensated for by the transmitter corrections as before. Utilization of these reference points could provide (a) independent determination of position and velocity and reduction of the geometrical dilution of the measurement errors, (b) measurement of the three orthogonal components in both position and velocity for limited coverage, and (c) an increase of coverage to 24 hours for two coordinates. The ability to determine three coordinates in both position and velocity may be of particular use when the concept is applied as an independent navigation system for surface, air, or spaceborne vehicles, but the extent of coverage still has to be investigated.

ACKNOWLEDGMENTS

The authors acknowledge the helpful discussions with Dr. S. Knowles on the error analysis and with Mr. D. Hammond on the special-purpose receiver design.

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Appendix A

ERROR ANALYSIS OF POSITION DETERMINATIONS

The effective error as a function of time and position can be derived from the basic relation of the geometric configuration of the earth-moon system (Fig. A1). Let the distance from the unknown location to the moon be expressed by

$$\rho = - b + \sqrt{s^2 + r^2 - 2\, sr \cos \beta} \,, \tag{A1}$$

where

b is the lunar radius at the point of reflection,

s the earth-moon center-to-center distance,

r the geocentric radius of the location, and

$\cos \beta = \sin \varphi \sin \delta + \cos \varphi \cos \delta \cos \gamma$.

In the latter equation,

φ is the geocentric latitude of the unknown location,

δ the moon's declination, and

γ the local hour angle $= \theta - \alpha - \lambda$,

where

θ is the Greenwich sidereal time,

α the moon's right ascension, and

λ the longitude of the unknown location.

If we assume the moon to be stationary, then the relative velocity of the point under observation is

$$\dot{\rho} = - \frac{sr}{\rho + b} \frac{d(\cos \beta)}{dt} = \frac{sr}{\rho + b} \omega_e \cos \varphi \cos \delta \sin \gamma \,, \tag{A2}$$

where $\omega_e = d\gamma/dt$ is the earth's rotation in radians per second.

Any error in longitude λ or latitude φ will produce an error in ρ and $\dot{\rho}$, which to first-order accuracy can be expressed as

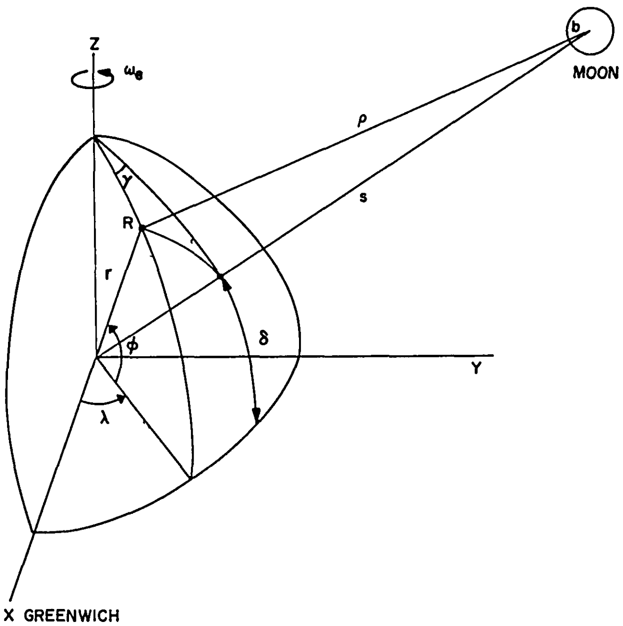

Fig. A1 - Geometry of the earth-moon system

$$\Delta\rho = - r \cos \varphi \cos \delta \sin \gamma \, \Delta\lambda - r(\cos \varphi \sin \delta - \sin \varphi \cos \delta \cos \gamma) \, \Delta\varphi \tag{A3}$$

and

$$\Delta\dot{\rho} = - r \, \omega_e \cos \varphi \cos \delta \cos \gamma \, \Delta\lambda - r \, \omega_e \sin \varphi \cos \delta \sin \gamma \, \Delta\varphi \; . \tag{A4}$$

Converting the angular errors to linear errors by setting $\Delta x \approx r \cos \varphi \, \Delta\lambda$ and $\Delta y = r \, \Delta\varphi$, the errors in position Δx and Δy are related to the differentials $\Delta\rho$ and $\Delta\dot{\rho}$ by Eqs. (A5) and (A6):

$$\frac{\Delta\rho}{\cos \delta} = - \sin \gamma \, \Delta x + (\cos \gamma \sin \varphi - \cos \varphi \tan \delta) \, \Delta y \tag{A5}$$

$$\frac{\Delta\dot{\rho}}{\omega_e \cos \delta} = - \cos \gamma \, \Delta x - \sin \gamma \sin \varphi \, \Delta y \; . \tag{A6}$$

It is seen that the effective measurement error is a weak function of the moon's declination and for extreme values of $\delta = \pm 28°$ will increase the basic measurement error by about 10 percent. To obtain the corrections of Δx and Δy, we have to solve Eqs. (A5) and (A6). Let

$$m_1 = \frac{\Delta\rho}{\cos\delta}, \qquad m_2 = \frac{\Delta\dot{\rho}}{\omega_e \cos\delta},$$

$$a_{11} = -\sin\gamma, \quad a_{12} = \sin\varphi\cos\gamma - \cos\varphi\tan\delta$$

$$a_{21} = -\cos\gamma, \quad a_{22} = -\sin\varphi\sin\gamma .$$

Then

$$\Delta x = \frac{\begin{vmatrix} m_1 & a_{12} \\ m_2 & a_{22} \end{vmatrix}}{\Delta} = \frac{m_1 a_{22} - m_2 a_{12}}{\Delta}, \tag{A7}$$

$$\Delta y = \frac{m_2 a_{11} - m_1 a_{21}}{\Delta}, \tag{A8}$$

where $\Delta = a_{11}a_{22} - a_{21}a_{12}$.

The variances of the corrections Δx and Δy in terms of the measurement variances $\sigma^2_{m_1}$ and $\sigma^2_{m_2}$ are given by

$$\sigma^2_{\Delta x} = \left(\frac{\partial \Delta x}{\partial m_1}\right)^2 \sigma^2_{m_1} + \left(\frac{\partial \Delta x}{\partial m_2}\right)^2 \sigma^2_{m_2}$$

$$= \frac{1}{\Delta^2}\left(a^2_{22}\,\sigma^2_{m_1} + a^2_{12}\,\sigma^2_{m_2}\right) \tag{A9}$$

$$\sigma^2_{\Delta y} = \frac{1}{\Delta^2}\left(a^2_{21}\,\sigma^2_{m_1} + a^2_{11}\,\sigma^2_{m_2}\right) \tag{A10}$$

For efficient operation, the measurement variances are equal: $\sigma^2_{m_1} = \sigma^2_{m_2} = \sigma^2_m$. Then the standard deviations of the corrections of Δx and Δy are

$$\sigma_{\Delta x} = \frac{1}{\Delta}\sqrt{a^2_{22} + a^2_{12}}\;\sigma_m \tag{A11}$$

and

$$\sigma_{\Delta y} = \frac{1}{\Delta}\sqrt{a^2_{21} + a^2_{11}}\;\sigma_m, \tag{A12}$$

where

$$\Delta = +\sin^2\gamma\sin\varphi + \cos\gamma^2\sin\varphi - \cos\gamma\cos\varphi\tan\delta$$

$$= \sin\varphi - \cos\gamma\cos\varphi\tan\delta,$$

$$\sqrt{a_{22}^2 + a_{12}^2} = (\sin^2 \varphi \sin^2 \gamma + \sin^2 \varphi \cos^2 \gamma$$

$$+ \cos^2 \varphi \tan^2 \delta$$

$$- 2 \sin \varphi \cos \varphi \tan \delta \cos \gamma)^{1/2} ,$$

and

$$\sqrt{a_{11}^2 + a_{21}^2} = \sqrt{\sin^2 \gamma + \cos^2 \gamma} = 1 .$$

Therefore,

$$\sigma_{\Delta x} = \frac{\sqrt{\sin^2 \varphi + \cos^2 \varphi \tan^2 \delta - \sin 2 \varphi \tan \delta \cos \gamma}}{\sin \varphi - \cos \gamma \cos \varphi \tan \delta} \sigma_m , \tag{A13}$$

$$\sigma_{\Delta y} = \frac{1}{\sin \varphi - \cos \gamma \cos \varphi \tan \delta} \sigma_m . \tag{A14}$$

These expressions will give the effective errors of Δx and Δy as a function of the latitude φ, the local hour angle γ, and the moon's declination δ. The computer printout in this appendix gives the program for solving Eqs. (A13) and (A14).

The equations are computed for various values of δ, φ, and γ in Tables A1 through A7 for $\sigma_m = 1/\cos \delta$. The tables marked "a" indicate the effective error in longitude, and those marked "b" give the corresponding error in latitude. The first vertical column gives the latitude of the observed position from +90° to -90° in 10° intervals. The top row gives the hour angle in 10° intervals (40 minutes of time) starting at transit. The tables extend symmetrically for negative hour angles (not shown). Each number, when multiplied by the given basic measurement error, indicates the effective error in meters. The tables are separated by about two days in time, corresponding to a change in declination by 9°. The effective errors increase as the latitude of the observed position approaches the declination of the moon.

Program for Solution of Eqs. (A13) and (A14)

```
TN5,4A                                                              09/03/68

      PROGRAM  COVERAGE
      DIMENSION  SIGX(19,7,19),SIGY(19,7,19),IPRINT(3)
      CCMMON/1/SIGX
      COMMON/2/SIGY
      TYPE INTEGER  DCLN,HA
      TYPE DOUBLE   DEC,PHI,GAMMA,TAND,SINP,COSP,COSG,DELTA,COSD,SIND
      TYPE DOUBLE  NUMERATR
      DATA (IPRINT = 8H(*+*,5X,,0,6HF4,1)))
  900 FORMAT (*1          ERROR IN LONGITUDE      FCR      DEC =*,I4,/)
  901 FORMAT (*0*,5X,*HA(+OR-) 0    10    20    30    40    50    60    70    80
     *    90  100  110  120  130  140  150  160  170  180*,/,*   PHI *)
  902 FORMAT (*+*,I5)
  903 FORMAT (*1          ERROR IN LATITUDE       FOR      DEC =*,I4,/)
  904 FORMAT (*0*)
  110 FORMAT (*5X,*,I2,*(1X*)
      DTR = 0.017453292519943296D
      DO 100   J=1,7
      DCLN = 9*(1-J) + 27
      DEC = DCLN*DTR
      COSD = DCOS(DEC)
      SIND = DSIN(DEC)
      TAND = SIND/COSD
      DO 50   I = 1,19
      LAT = 10*(1 - I) + 90
      PHI = LAT*DTR
      SINP = DSIN(PHI)
      COSP = DCOS(PHI)
      DO 30   K = 1,19
      HA = 10*(K - 1)
      GAMMA = HA*DTR
      COSG = DCOS(GAMMA)
      IF(SINP*SIND + COSP*COSD*COSG ,GT, 0)  GO TO 10
      SIGX(I,J,K) = SIGY(I,J,K) = 0.
      GO TO 30
   10 IF(LAT ,NE, DCLN  ,OR,  HA ,NE, 0)   GO TO 20
      COSG = DCOS(0.1*DTR)
   20 DELTA = COSP*TAND-COSG - SINP
      NUMERATR = DSQRT(SINP**2 + (COSP*TAND)**2 - 2*SINP*COSP*TAND*COSG)
      IF(DELTA ,EQ, 0)   DELTA = 1,0D-30
      IF(NUMERATR ,EQ, 0,)   NUMERATR = 1,0D-30
      SIGX(I,J,K) = NUMERATR/DELTA
      SIGY(I,J,K) = DABS(1,/DELTA)/COSD
      SIGX(I,J,K) = ABSF(SIGX(I,J,K))/COSD
   30 CONTINUE
   50 CONTINUE
      PRINT 900,  DCLN
      PRINT 901
      DO 35   I = 1,19
      LAT = 10*(1 - I) + 90
      PRINT 902, LAT
      M = 0
      DO 33  K=1,19
      IF(SIGX(I,J,K) ,EQ, 0,)  GO TO 34
   33 M = M + 1
   34 IF(M ,EQ, 0)  GO TO 35
      ENCODE (8,110,IPRINT(2)) M
```

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```
      PRINT IPRINT,       (SIGX(I,J,K),K=1,M)
   35 PRINT 904
      PRINT 903,  DCLN
      PRINT 901
      DO 40   I = 1,19
      LAT = 10*(1 - I) + 90
      PRINT 902, LAT
      M = 0
      DO 38  K=1,19
      IF(SIGY(I,J,K) .EQ. 0.)  GO TO 39
   38 M = M + 1
   39 IF(M .EQ. 0)  GO TO 40
      ENCODE (8,110,IPRINT(2)) M
      PRINT IPRINT,       (SIGY(I,J,K),K=1,M)
   40 PRINT 904
  100 CONTINUE
      END
```

Table A1a
Error in Longitude for a Declination Angle of 27°

ERROR IN LONGITUDE FOR DEC = 27

PHI \ HA(+OR-)	0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180
90	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1
80	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1
70	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1
60	1.1	1.1	1.1	1.1	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.1	1.1	1.1	1.1			
50	1.1	1.1	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2						
40	1.1	1.2	1.2	1.3	1.4	1.4	1.4	1.4	1.4	1.3	1.3	1.2							
30	1.1	1.7	2.3	2.4	2.3	2.1	1.9	1.7	1.6	1.5	1.4								
20	1.1	1.3	2.0	3.9	14.0	12.1	4.7	3.0	2.3	1.9	1.7								
10	1.1	1.2	1.3	1.6	2.1	3.1	6.4	****	6.5	3.4									
0	1.1	1.1	1.2	1.3	1.5	1.7	2.2	3.3	6.5	****									
-10	1.1	1.1	1.2	1.2	1.3	1.4	1.6	1.9	2.4										
-20	1.1	1.1	1.1	1.2	1.2	1.3	1.4	1.5											
-30	1.1	1.1	1.1	1.2	1.2	1.2	1.3	1.3											
-40	1.1	1.1	1.1	1.1	1.2	1.2	1.2												
-50	1.1	1.1	1.1	1.1	1.1	1.2													
-60	1.1	1.1	1.1																
-70																			
-80																			
-90																			

Table A1b
Error in Latitude for a Declination Angle of 27°

ERROR IN LATITUDE FOR DEC = 27

PHI \ HA(+OR-)	0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180
90	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1
80	1.3	1.3	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.0	1.0
70	1.5	1.5	1.4	1.4	1.4	1.4	1.3	1.3	1.2	1.2	1.2	1.1	1.1	1.1	1.0	1.0	1.0	1.0	1.0
60	1.8	1.8	1.8	1.7	1.7	1.6	1.5	1.4	1.4	1.3	1.2	1.2	1.1	1.1	1.1	1.0			
50	2.6	2.5	2.4	2.3	2.2	2.0	1.9	1.7	1.6	1.5	1.4	1.3	1.2						
40	4.4	4.3	4.1	3.7	3.3	2.9	2.5	2.2	2.0	1.7	1.6	1.4							
30	19.1	17.1	13.1	9.5	6.9	5.2	4.0	3.2	2.7	2.2	1.9								
20	8.2	8.7	10.4	15.5	45.3	32.8	10.9	6.3	4.3	3.3	2.6								
10	3.4	3.5	3.8	4.3	5.3	7.5	14.5	****	13.0	6.5									
0	2.2	2.2	2.3	2.5	2.9	3.4	4.4	6.4	12.7	****									
-10	1.7	1.7	1.7	1.8	2.0	2.3	2.6	3.3	4.3										
-20	1.4	1.4	1.4	1.5	1.6	1.7	1.9	2.2											
-30	1.2	1.2	1.2	1.3	1.3	1.4	1.6	1.7											
-40	1.1	1.1	1.1	1.1	1.2	1.3	1.3												
-50	1.0	1.0	1.0	1.1	1.1	1.1													
-60	1.0	1.0	1.0																
-70																			
-80																			
-90																			

Table A2a
Error in Longitude for a Declination Angle of 18°

ERROR IN LONGITUDE FOR DEC = 18

HA(+OR-) PHI	0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180
90	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1.1	1,1	1,1	1,1	1,1	1,1
80	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1
70	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1			
60	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1						
50	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1							
40	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1								
30	1,1	1,1	1,1	1,2	1,2	1,3	1,3	1,3	1,2	1,2	1,2								
20	1,1	1,7	2,3	2,3	2,2	2,0	1,8	1,6	1,5	1,4									
10	1,1	1,1	1,4	1.9	3,2	8,1	21.4	5,0	3,0	2,2									
0	1,1	1,1	1,1	1,2	1,4	1.6	2,1	3,1	6,1	****									
-10	1,1	1,1	1,1	1,1	1,2	1,3	1,4	1,5	1,8										
-20	1,1	1,1	1,1	1,1	1,1	1,1	1,2	1,2	1,3										
-30	1,1	1,1	1,1	1,1	1,1	1.1	1.1	1,2											
-40	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1											
-50	1,1	1,1	1,1	1,1	1,1	1,1	1,1												
-60	1,1	1,1	1,1	1.1	1,1	1,1													
-70	1,1	1.1	1,1																
-80																			
-90																			

Table A2b
Error in Latitude for a Declination Angle of 18°

ERROR IN LATITUDE FOR DEC = 18

PHI \ HA(+OR-)	0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180
90	1,1	1,1	1,1	1,1	1,1	1,1	1.1	1,1	1.1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1
80	1,1	1,1	1,1	1.1	1,1	1,1	1,1	1,1	1.1	1,1	1,1	1,0	1,0	1.0	1,0	1,0	1,0	1,0	1,0
70	1,3	1,3	1,3	1,2	1,2	1.2	1,2	1,2	1.1	1,1	1,1	1,1	1,1	1,0	1,0	1,0			
60	1,5	1,5	1,5	1,4	1,4	1,4	1,3	1,3	1.3	1,2	1,2	1,1	1,1						
50	1,9	1,9	1,8	1,8	1,7	1,7	1,6	1,5	1.4	1,4	1,3	1,3							
40	2,7	2,6	2,6	2,5	2,3	2,2	2,0	1,9	1,8	1,6	1,5								
30	4,8	4,7	4,5	4.1	3,7	3,3	2,9	2,6	2.3	2,1	1,9								
20	28,7	25,4	19,1	13,5	9,7	7,2	5,6	4,4	3,6	3,1									
10	7,2	7,4	8,3	10,2	14,7	32,8	77,0	16,4	8,9	6,1									
0	3,2	3,3	3,4	3.7	4,2	5,0	6,5	9,5	18,6	****									
-10	2,1	2,2	2,2	2,3	2,5	2,8	3,2	3,7	4.6										
-20	1,6	1,6	1,7	1,7	1,8	2,0	2,1	2,4	2.7										
-30	1,3	1,4	1,4	1,4	1,5	1,5	1,6	1,8											
-40	1,2	1,2	1,2	1.2	1,3	1,3	1,4	1,4											
-50	1,1	1,1	1,1	1,1	1,1	1,2	1,2												
-60	1,0	1,0	1,0	1,0	1,1	1,1													
-70	1,0	1,0	1,0																
-80																			
-90																			

Table A3a
Error in Longitude for a Declination Angle of 9°

ERROR IN LONGITUDE FOR DEC = 9

PHI \ HA(+OR-)	0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180
90	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
80	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0			
70	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0							
60	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0								
50	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0								
40	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0									
30	1.0	1.0	1.0	1.0	1.0	1.0	1.1	1.1	1.1	1.0									
20	1.0	1.0	1.0	1.1	1.1	1.1	1.1	1.1	1.1	1.1									
10	1.0	1.7	2.2	2.3	2.1	1.9	1.8	1.6	1.5	1.4									
0	1.0	1.0	1.1	1.2	1.3	1.6	2.0	3.0	5.8	****									
-10	1.0	1.0	1.0	1.0	1.1	1.1	1.1	1.2	1.3										
-20	1.0	1.0	1.0	1.0	1.0	1.0	1.1	1.1	1.1										
-30	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0										
-40	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0										
-50	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0											
-60	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0											
-70	1.0	1.0	1.0	1.0	1.0	1.0	1.0												
-80	1.0	1.0	1.0																
-90																			

Table A3b
Error in Latitude for a Declination Angle of 9°

ERROR IN LATITUDE FOR DEC = 9

PHI \ HA(+0R=)	0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180
90	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0
80	1,1	1,1	1,1	1,1	1,1	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0			
70	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1							
60	1,3	1,3	1,3	1,3	1,3	1,2	1,2	1,2	1,2	1,2	1,2								
50	1,5	1,5	1,5	1,5	1,5	1,4	1,4	1,4	1,4	1,3	1,3								
40	1,9	1,9	1,9	1,9	1,8	1,8	1,7	1,7	1,6	1,6									
30	2,8	2,8	2,7	2,7	2,6	2,5	2,3	2,2	2,1	2,0									
20	5,2	5,2	5,0	4,8	4,4	4,1	3,8	3,5	3,2	3,0									
10	57,3	50,5	37,4	26,3	18,7	13,8	10,6	8,4	6,9	5,8									
0	6,4	6,5	6,8	7,4	8,3	9,9	12,8	18,7	36,8	****									
=10	3,1	3,1	3,2	3,3	3,5	3,7	4,0	4,5	5,0										
=20	2,1	2,1	2,1	2,2	2,2	2,3	2,4	2,6	2,8										
=30	1,6	1,6	1,6	1,6	1,7	1,7	1,8	1,9	1,9										
=40	1,3	1,3	1,3	1,4	1,4	1,4	1,4	1,5	1,5										
-50	1,2	1,2	1,2	1,2	1,2	1,2	1,2	1,3											
=60	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1											
=70	1,0	1,0	1,0	1,0	1,0	1,0	1,0												
=80	1,0	1,0	1,0																
-90																			

Table A4a

Error in Longitude for a Declination Angle of 0°

ERROR IN LONGITUDE FOR DEC = 0

PHI \ HA(+OR-)	0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180
90	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0									
80	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0									
70	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0									
60	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0									
50	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0									
40	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0									
30	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0									
20	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0									
10	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0									
0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0									
-10	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0									
-20	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0									
-30	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0									
-40	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0									
-50	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0									
-60	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0									
-70	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0									
-80	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0									
-90	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0									

Table A4b
Error in Latitude for a Declination Angle of 0°

ERROR IN LATITUDE FOR DEC = 0

PHI \ HA(+OR-)	0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180
90	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0									
80	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0									
70	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1									
60	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2									
50	1.3	1.3	1.3	1.3	1.3	1.3	1.3	1.3	1.3	1.3									
40	1.6	1.6	1.6	1.6	1.6	1.6	1.6	1.6	1.6	1.6									
30	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0									
20	2.9	2.9	2.9	2.9	2.9	2.9	2.9	2.9	2.9	2.9									
10	5.8	5.8	5.8	5.8	5.8	5.8	5.8	5.8	5.8	5.8									
0	****	****	****	****	****	****	****	****	****	****									
-10	5.8	5.8	5.8	5.8	5.8	5.8	5.8	5.8	5.8	5.8									
-20	2.9	2.9	2.9	2.9	2.9	2.9	2.9	2.9	2.9	2.9									
-30	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0	2.0									
-40	1.6	1.6	1.6	1.6	1.6	1.6	1.6	1.6	1.6	1.6									
-50	1.3	1.3	1.3	1.3	1.3	1.3	1.3	1.3	1.3	1.3									
-60	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2									
-70	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1									
-80	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0									
-90	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0									

Table A5a
Error in Longitude for a Declination Angle of -9°

ERROR IN LONGITUDE FOR DEC = -9

PHI \ HA(+OR-)	0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180
90																			
80	1.0	1.0	1.0																
70	1.0	1.0	1.0	1.0	1.0	1.0	1.0												
60	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0											
50	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0											
40	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0										
30	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0										
20	1.0	1.0	1.0	1.0	1.0	1.0	1.1	1.1	1.1										
10	1.0	1.0	1.0	1.0	1.1	1.1	1.1	1.2	1.3										
0	1.0	1.0	1.1	1.2	1.3	1.6	2.0	3.0	5.8	****									
-10	1.0	1.7	2.2	2.3	2.1	1.9	1.8	1.6	1.5	1.4									
-20	1.0	1.0	1.0	1.1	1.1	1.1	1.1	1.1	1.1	1.1									
-30	1.0	1.0	1.0	1.0	1.0	1.0	1.1	1.1	1.1	1.0									
-40	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0									
-50	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0								
-60	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0								
-70	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0							
-80	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0			
-90	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0

Table A5b
Error in Latitude for a Declination Angle of -9°

ERROR IN LATITUDE FOR DEC = -9

PHI \ HA(+OR-)	0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180
90																			
80	1,0	1,0	1,0																
70	1,0	1,0	1,0	1,0	1,0	1,0	1,0												
60	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1											
50	1,2	1,2	1,2	1,2	1,2	1,2	1,2	1,3											
40	1,3	1,3	1,3	1,4	1,4	1,4	1,4	1,5	1,5										
30	1,6	1,6	1,6	1,6	1,7	1,7	1,8	1,9	1,9										
20	2,1	2,1	2,1	2,2	2,2	2,3	2,4	2,6	2,8										
10	3,1	3,1	3,2	3,3	3,5	3,7	4,0	4,5	5,0										
0	6,4	6,5	6,8	7,4	8,3	9,9	12,8	18,7	36,8	****									
-10	57,3	50,5	37,4	26,3	18,7	13,8	10,6	8,4	6,9	5,8									
-20	5,2	5,2	5,0	4,8	4,4	4,1	3,8	3,5	3,2	3,0									
-30	2,8	2,8	2,7	2,7	2,6	2,5	2,3	2,2	2,1	2,0									
-40	1,9	1,9	1,9	1,9	1,8	1,8	1,7	1,7	1,6	1,6									
-50	1,5	1,5	1,5	1,5	1,5	1,4	1,4	1,4	1,4	1,3	1,3								
-60	1,3	1,3	1,3	1,3	1,3	1,2	1,2	1,2	1,2	1,2	1,2								
-70	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1							
-80	1,1	1,1	1,1	1,1	1,1	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0			
-90	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0

Table A6a
Error in Longitude for a Declination Angle of -18°

ERROR IN LONGITUDE FOR DEC = -18

PHI \ HA(+OR-)	0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180
90																			
80																			
70	1,1	1,1	1,1																
60	1,1	1,1	1,1	1,1	1,1	1,1													
50	1,1	1,1	1,1	1,1	1,1	1,1	1,1												
40	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1											
30	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,2											
20	1,1	1,1	1,1	1,1	1,1	1,1	1,2	1,2	1.3										
10	1,1	1,1	1,1	1,1	1,2	1,3	1,4	1,5	1,8										
0	1,1	1,1	1,1	1,2	1,4	1,6	2,1	3,1	6,1	****									
-10	1,1	1,1	1,4	1,9	3,2	8,1	21,4	5,0	3,0	2,2									
-20	1,1	1,7	2,3	2,3	2,2	2,0	1,8	1,6	1,5	1,4									
-30	1,1	1,1	1,1	1,2	1,2	1,3	1,3	1,3	1,2	1,2	1,2								
-40	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1								
-50	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1							
-60	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1						
-70	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1			
-80	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1
-90	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1

Table A6b
Error in Latitude for a Declination Angle of -18°

ERROR IN LATITUDE FOR DEC = -18

PHI \ HA(+EW=)	0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180
90																			
80																			
70	1,0	1.0	1.0																
60	1,0	1,0	1.0	1,0	1,1	1,1													
50	1,1	1,1	1.1	1,1	1,1	1.2	1,2												
40	1,2	1,2	1.2	1.2	1,3	1.3	1,4	1,4											
30	1,3	1 4	1.4	1,4	1.5	1.5	1,6	1,8											
20	1,6	1.6	1.7	1.7	1,8	2,0	2,1	2,4	2.7										
10	2,1	2,2	2,2	2,3	2,5	2,8	3,2	3,7	4.6										
0	3,2	3,3	3,4	3,7	4,2	5,0	6,5	9,5	18,6	****									
-10	7,2	7,4	8,3	10,2	14,7	32,8	77,0	16,4	8,9	6,1									
-20	28,7	25,4	19,1	13,5	9,7	7,2	5,6	4,4	3,6	3,1									
-30	4,8	4,7	4,5	4,1	3,7	3,3	2,9	2,6	2,3	2,1	1,9								
-40	2,7	2,6	2,6	2,5	2,3	2,2	2,0	1,9	1,8	1,6	1,5								
-50	1,9	1,9	1,8	1,8	1,7	1,7	1,6	1,5	1,4	1,4	1,3	1,3							
-60	1,5	1,5	1,5	1,4	1,4	1,4	1,3	1,3	1,3	1,2	1,2	1,1	1,1						
-70	1,3	1,3	1,3	1,2	1,2	1,2	1,2	1,2	1,1	1,1	1,1	1,1	1,1	1,0	1,0	1,0			
-80	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,0	1,0	1,0	1,0	1,0	1,0	1,0	1,0
-90	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1

Table A7a
Error in Longitude for a Declination Angle of -27°

ERROR IN LONGITUDE FOR DEC = -27

HA(+OR-) PHI	0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180
90																			
80																			
70																			
60	1.1	1.1	1.1																
50	1.1	1.1	1.1	1.1	1.1	1.2													
40	1.1	1.1	1.1	1.1	1.2	1.2	1.2												
30	1.1	1.1	1.1	1.2	1.2	1.2	1.3	1.3											
20	1.1	1.1	1.1	1.2	1.2	1.3	1.4	1.5											
10	1.1	1.1	1.2	1.2	1.3	1.4	1.6	1.9	2.4										
0	1.1	1.1	1.2	1.3	1.5	1.7	2.2	3.3	6.5	****									
-10	1.1	1.2	1.3	1.6	2.1	3.1	6.4	****	6.5	3.4									
-20	1.1	1.3	2.0	3.9	14.0	12.1	4.7	3.0	2.3	1.9	1.7								
-30	1.1	1.7	2.3	2.4	2.3	2.1	1.9	1.7	1.6	1.5	1.4								
-40	1.1	1.2	1.2	1.3	1.4	1.4	1.4	1.4	1.4	1.3	1.3	1.2							
-50	1.1	1.1	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2						
-60	1.1	1.1	1.1	1.1	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.1	1.1	1.1	1.1			
-70	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1
-80	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1
-90	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1

Table A7b
Error in Latitude for a Declination Angle of -27°

ERROR IN LATITUDE FOR DEC = -27

PHI \ HA(+OR-)	0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180
90																			
80																			
70																			
60	1,0	1,0	1,0																
50	1,0	1,0	1,0	1,1	1,1	1,1													
40	1,1	1,1	1,1	1,1	1,2	1,3	1,3												
30	1,2	1,2	1,2	1,3	1,3	1,4	1,6	1,7											
20	1,4	1,4	1,4	1,5	1,6	1,7	1,9	2,2											
10	1,7	1,7	1,7	1,8	2,0	2,3	2,6	3,3	4,3										
0	2,2	2,2	2,3	2,5	2,9	3,4	4,4	6,4	12,7	****									
-10	3,4	3,5	3,8	4,3	5,3	7,5	14,5	****	13,0	6,5									
-20	8,2	8,7	10,4	15,5	45,3	32,8	10,9	6,3	4,3	3,3	2,6								
-30	19,1	17,1	13,1	9,5	6,9	5,2	4,0	3,2	2,7	2,2	1,9								
-40	4,4	4,3	4,1	3,7	3,3	2,9	2,5	2,2	2,0	1,7	1,6	1,4							
-50	2,6	2,5	2,4	2,3	2,2	2,0	1,9	1,7	1,6	1,5	1,4	1,3	1,2						
-60	1,8	1,8	1,8	1,7	1,7	1,6	1,5	1,4	1,4	1,3	1,2	1,2	1,1	1,1	1,1	1,0			
-70	1,5	1,5	1,4	1,4	1,4	1,4	1,3	1,3	1,2	1,2	1,2	1,1	1,1	1,1	1,0	1,0	1,0	1,0	1,0
-80	1,3	1,3	1,2	1,2	1,2	1,2	1,2	1,2	1,2	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,0	1,0
-90	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1	1,1

Appendix B

EFFECT OF SYSTEMATIC ERRORS IN SHIP'S VELOCITY ON POSITION DETERMINATION

When the location of a moving ship is to be determined, it is necessary to include in the computation the doppler shift and position shift caused by the ship's motion relative to the moon. The error in the ship's velocity, if excessive, will be directly reflected in an error of position. However, by taking measurements over several hours, it is possible to reduce the errors in both position and velocity to the basic measurement error.

To show the required computation as well as the functional relation between position and velocity, the observation equation will be derived. The range and range rate including the ship's velocity are given as a function of time for an assumed stationary moon by

$$\rho(t) \approx \sqrt{s^2 + r^2 - 2\,sr\,\cos\beta(t)} + \int_{t_0}^{t} v_{sr}\,dt \tag{B1}$$

$$\dot{\rho}(t) \approx -r\frac{d(\cos\beta)}{dt} + v_{sr}\,, \tag{B2}$$

where v_{sr} is the radial velocity of the ship relative to the moon and the other symbols are as defined in Appendix A. For a short observation interval, the coordinates of the moon and the position of the ship are effectively stationary, and it is further assumed that the values of ρ and $\dot{\rho}$ are referred to the midpoint of the observational interval T_0, or $(t - t_0)/2 = T_0/2$. Then

$$\rho_{T_0/2} \approx \sqrt{s^2 + r^2 - 2\,sr\,\cos\beta} + v_{sr}\,T_0/2 \tag{B3}$$

$$\dot{\rho}_{T_0/2} \approx -r\left.\frac{d(\cos\beta)}{dt}\right|_{T_0/2} + v_{sr}\,. \tag{B4}$$

First, v_{sr} will be derived in Greenwich equatorial coordinates. Assume a coordinate system with the x axis pointing to the Greenwich meridian and the z axis to the north pole as shown in Fig. B1. Any point S on the earth's surface can be expressed in terms of the earth's radius r, the latitude φ, and the longitude λ (λ is defined as the negative of the conventional astronomical longitude).

The moon's position can be similarly described where the moon's distance is s, the latitude φ' is expressed as the declination δ, and the longitude λ' is given by the difference between the right ascension α and the Greenwich sidereal time θ. Thus $\lambda' = \alpha - \theta$.

The radial velocity is obtained from

$$v_{sr} = -\,\mathbf{v}_s \cdot \hat{\rho} \approx \frac{-\mathbf{v}_s \cdot \mathbf{s}}{s}\,, \tag{B5}$$

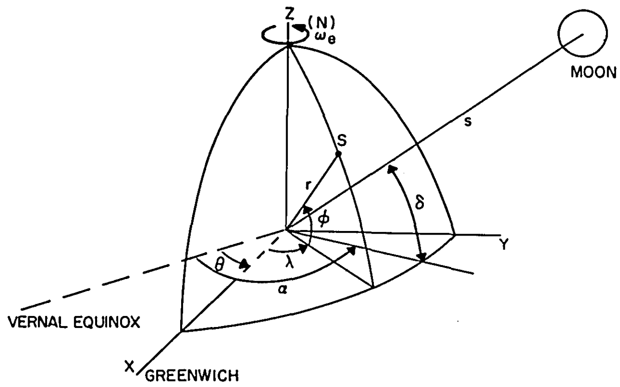

Fig. B1 - Geometry of the earth-moon system including celestial coordinates

where $\mathbf{v}_s$ is the ship's velocity vector, and the negative sign indicates that the radial velocity is measured relative to the center of the moon.

To determine the vector $\mathbf{v}_s$ in equatorial coordinates, first define $\mathbf{v}_s$ in a local coordinate system at the point of observation, with the y'' axis pointing east and the z'' axis pointing north. The x'' axis is normal to the y'' and z'' axes, and forms a right-handed coordinate system. The direction of $\mathbf{v}_s$ is measured by the angle ζ from east to north as shown in Fig. B2.

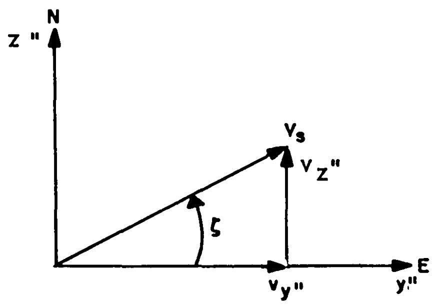

Fig. B2 - Components of ship's velocity

The components of $\mathbf{v}_s$ are then given by

$$v_{x''} = 0 \ ,$$

$$v_{y''} = v_s \cos \zeta \ ,$$

$$v_{z''} = v_s \sin \zeta \ .$$

To convert to the Greenwich equatorial coordinate system, two rotations are required. First rotate about the y'' axis by φ. This rotation brings the x'' axis parallel to the equatorial plane and the z'' axis parallel to the earth's axis of rotation. A second rotation by $-\lambda$ about the z' axis brings the x' axis to the Greenwich meridian and makes the y' axis parallel to the y axis in Fig. B1. Using the transformation equation as given by Brouwer and Clemence,* we obtain from the first rotation about y'' axis by φ

$$v_{x'} = v_{x''} \cos\varphi - v_{z''} \sin\varphi = - v_{z''} \sin\varphi \,,$$

$$v_{y'} = v_{y''} \,,$$

$$v_{z'} = v_{x''} \sin\varphi + v_{z''} \cos\varphi = v_{z''} \cos\varphi \,,$$

and from the second rotation about z' axis by $-\lambda$

$$v_x = v_{x'} \cos\lambda - v_{y'} \sin\lambda \,,$$

$$v_y = v_{x'} \sin\lambda + v_{y'} \cos\lambda \,,$$

$$v_z = v_{z'} \,,$$

giving finally the velocity vector in the Greenwich-equatorial coordinate system as

$$\begin{aligned} \mathbf{v}_s = & - i\,(v_{y''} \sin\lambda + v_{z''} \sin\varphi \cos\lambda) \\ & + j\,(v_{y''} \cos\lambda - v_{z''} \sin\varphi \sin\lambda) \\ & + k\, v_{z''} \cos\varphi \end{aligned} \tag{B6}$$

in terms of the east and north velocities $v_{y''}$ and $v_{z''}$.

The lunar distance vector is

$$\mathbf{s} = s\,[i \cos(\alpha - \theta) \cos\delta + j \sin(\alpha - \theta) \cos\delta + k \sin\delta] \,. \tag{B7}$$

The velocity of the ship relative to the moon is then given by

$$\begin{aligned} v_{sr} &= \frac{-\mathbf{v} \cdot \mathbf{s}}{s} \\ &= \cos\delta\,[\sin\gamma\, v_{y''} + (\sin\varphi \cos\gamma - \tan\delta \cos\varphi)\, v_{z''}] \,, \end{aligned} \tag{B8}$$

where γ is the local hour angle of the moon, i.e., $\gamma = \alpha - \theta - \lambda$ and λ is now defined by astronomical convention.

The range and range rate equations can then be expressed in Greenwich equatorial coordinates as

$$\begin{aligned} \rho \approx & \sqrt{s^2 + r^2 - 2\,sr\cos\beta} \\ & + \frac{T_0}{2} \cos\delta\,[\sin\gamma\, v_{y''} + (\sin\varphi \cos\gamma - \tan\delta \cos\varphi)\, v_{z''}] \end{aligned} \tag{B9}$$

---

*D. Brouwer and G. M. Clemence, "Methods of Celestial Mechanics," New York:Academic Press, p. 40, 1961.

and

$$\dot{\rho} \approx \cos\delta\,[r\omega_e \cos\varphi \sin\gamma + \sin\gamma\, v_{y''} + (\sin\varphi\cos\gamma - \tan\delta\cos\varphi)\, v_{z''}] \,. \tag{B10}$$

The differential corrections are obtained from

$$\Delta\rho = \frac{\partial\rho}{\partial\lambda}\Delta\lambda + \frac{\partial\rho}{\partial\varphi}\Delta\varphi + \frac{\partial\rho}{\partial v_{y''}}\Delta v_{y''} + \frac{\partial\rho}{\partial v_{z''}}\Delta v_{z''} \,,$$

$$\Delta\dot{\rho} = \frac{\partial\dot{\rho}}{\partial\lambda}\Delta\lambda + \frac{\partial\dot{\rho}}{\partial\varphi}\Delta\varphi + \frac{\partial\dot{\rho}}{\partial v_{y''}}\Delta v_{y''} + \frac{\partial\dot{\rho}}{\partial v_{z''}}\Delta v_{z''} \,.$$

The eight coefficients are given to first-order accuracy (assume $(T_0 v_{y''})/2r$ and $(T_0 v_{z''})/2r \ll 1$ and $(T_0/2)\,\Delta v_{y''}$, $(T_0/2)\,\Delta v_{z''} < \sigma_m$);

$$\frac{\partial\rho}{\partial\lambda} = -\, r\cos\varphi\cos\delta\sin\gamma$$

$$\frac{\partial\rho}{\partial\varphi} = r\cos\delta\,(\sin\varphi\cos\gamma - \cos\varphi\tan\delta)$$

$$\frac{\partial\rho}{\partial v_{y''}} \approx 0$$

$$\frac{\partial\rho}{\partial v_{z''}} \approx 0$$

$$\frac{\partial\dot{\rho}}{\partial\lambda} = -\cos\delta\,(r\omega_e\cos\varphi\cos\gamma + \cos\gamma\, v_{y''} - \sin\varphi\sin\gamma\, v_{z''})$$

$$\frac{\partial\dot{\rho}}{\partial\varphi} = -\cos\delta\,[r\omega_e\sin\varphi\sin\gamma - (\cos\varphi\cos\gamma + \tan\delta\sin\varphi)\, v_{z''}]$$

$$\frac{\partial\dot{\rho}}{\partial v_{y''}} = \sin\gamma\cos\delta$$

$$\frac{\partial\dot{\rho}}{\partial v_{z''}} = \cos\delta\,(\sin\varphi\cos\gamma - \cos\varphi\tan\delta) \,.$$

After converting to linear dimensions,

$$\Delta x = r\cos\varphi\,\Delta\lambda \,,$$

$$\Delta y = r\,\Delta\varphi \,,$$

the observation equations become

$$\frac{\Delta\rho}{\cos\delta} = -\sin\gamma\,\Delta x + (\sin\varphi\cos\gamma - \cos\varphi\tan\delta)\,\Delta y\ , \tag{B11}$$

$$\frac{\Delta\dot{\rho}}{\omega_e\cos\delta} = \left[-\cos\gamma\left(\frac{v_{y''}}{r\omega_e\cos\varphi} + 1\right) + \sin\gamma\sin\varphi\,\frac{v_{z''}}{r\omega_e\cos\varphi}\right]\Delta x$$

$$+ \left[-\sin\varphi\sin\gamma + (\cos\varphi\cos\gamma + \tan\delta\sin\varphi)\,\frac{v_{z''}}{r\omega_e}\right]\Delta y$$

$$+ \sin\gamma\,\frac{\Delta v_{y''}}{\omega_e} + (\sin\varphi\cos\gamma - \cos\varphi\tan\delta)\,\frac{\Delta v_{z''}}{\omega_e}\ . \tag{B12}$$

For the moderate velocities of ships $v_{y''}/(r\omega_e\cos\varphi)$, $v_{z''}/(r\omega_e\cos\varphi)$, and $v_{z''}/(r\omega_e) \ll 1$, and by rearranging the equations for solution of Δx and Δy, we obtain

$$\frac{\Delta\rho}{\cos\delta} = -\sin\gamma\,\Delta x + (\cos\gamma\sin\varphi - \cos\varphi\tan\delta)\,\Delta y \tag{B13}$$

and

$$\frac{\Delta\dot{\rho}}{\omega_e\cos\delta} - \sin\gamma\,\frac{\Delta v_{y''}}{\omega_e} - (\sin\varphi\cos\gamma - \cos\varphi\tan\delta)\,\frac{\Delta v_{z''}}{\omega_e}$$

$$= -\cos\gamma\,\Delta x - \sin\gamma\sin\varphi\,\Delta y\ . \tag{B14}$$

Let

$$m_1 = \frac{\Delta\rho}{\cos\delta}\ ;$$

$$m_2' = \frac{\Delta\dot{\rho}}{\omega_e\cos\delta} - \sin\gamma\,\frac{\Delta v_{y''}}{\omega_e} - (\cos\gamma\sin\varphi - \cos\varphi\tan\delta)\,\frac{\Delta v_{z''}}{\omega_e}\ ,$$

and

$$a_{11} = -\sin\gamma\ ,$$

$$a_{12} = \cos\gamma\sin\varphi - \cos\varphi\tan\delta\ ,$$

$$a_{21} = -\cos\gamma\ ,$$

$$a_{22} = -\sin\gamma\sin\varphi\ ;$$

then

$$\Delta x = \frac{\begin{vmatrix} m_1 & a_{12} \\ m_2' & a_{22} \end{vmatrix}}{\Delta} = \frac{m_1\,a_{22} - m_2'\,a_{12}}{\Delta}\ ,$$

$$\Delta y = \frac{\begin{vmatrix} a_{11} & m_1 \\ a_{21} & m_2' \end{vmatrix}}{\Delta} = \frac{m_2' \, a_{11} - m_1 \, a_{21}}{\Delta} ,$$

where $\Delta = a_{11}a_{22} - a_{12}a_{21} = \sin \varphi - \cos \gamma \cos \varphi \tan \delta$, or

$$\Delta x = \frac{1}{\Delta}\left(- \sin \gamma \sin \varphi \frac{\Delta \rho}{\cos \delta} - a_{12} \frac{\Delta \dot{\rho}}{\omega_e \cos \delta} + a_{12} \sin \gamma \frac{\Delta v_{y''}}{\omega_e} + a_{12}^2 \frac{\Delta v_{z''}}{\omega_e}\right), \tag{B15}$$

$$\Delta y = \frac{1}{\Delta}\left(- \sin \gamma \frac{\Delta \dot{\rho}}{\omega_e \cos \delta} + \sin^2 \gamma \frac{\Delta v_{y''}}{\omega_e} + \sin \gamma \, a_{12} \frac{\Delta v_{z''}}{\omega_e} + \cos \gamma \frac{\Delta \rho}{\cos \delta}\right) \tag{B16}$$

The dependence of the variance of Δx and Δy on the variance of $\Delta v_{y''}$ and $\Delta v_{z''}$ is found from

$$\sigma^2_{\Delta x_v} = \left\{\left[\frac{\partial \Delta x}{\partial\left(\frac{\Delta v_{y''}}{\omega_e}\right)}\right]^2 \frac{\sigma^2_{\Delta v_{y''}}}{\omega_e} + \frac{\partial \Delta x}{\partial\left(\frac{\Delta v_{z''}}{\omega_e}\right)} \frac{\sigma^2_{\Delta v_{z''}}}{\omega_e}\right\}$$

and

$$\sigma^2_{\Delta y_v} = \left\{\left[\frac{\partial \Delta y}{\partial\left(\frac{\Delta y_{y''}}{\omega_e}\right)}\right]^2 \frac{\sigma^2_{\Delta v_y}}{\omega_e} + \left[\frac{\partial \Delta y}{\partial\left(\frac{\Delta v_{z''}}{\omega_e}\right)}\right]^2 \frac{\sigma^2_{\Delta v_{z''}}}{\omega_e}\right\}$$

Letting

$$\frac{\sigma^2_{\Delta v_{y''}}}{\omega_e} = \frac{\sigma^2_{\Delta v_{z''}}}{\omega_e} = \sigma_m$$

and substituting for

$$\frac{\partial \Delta x}{\partial\left(\frac{\Delta v_{y''}}{\omega_e}\right)} = \frac{1}{\Delta} a_{12} \sin \gamma , \qquad \frac{\partial \Delta x}{\partial\left(\frac{\Delta v_{z''}}{\omega_e}\right)} = \frac{1}{\Delta} a_{12}^2 ,$$

$$\frac{\partial \Delta y}{\partial \left(\frac{\Delta v_y''}{\omega_e}\right)} = \frac{1}{\Delta} \sin^2 \gamma \ , \qquad \frac{\partial \Delta y}{\partial \left(\frac{\Delta v_z''}{\omega_e}\right)} = \frac{\sin \gamma \ a_{12}}{\Delta} ,$$

the standard deviations of Δx and Δy are given by

$$\sigma_{\Delta x_v} = \frac{1}{\Delta} \left(a_{12}^2 \sin^2 \gamma + a_{12}^4 \right)^{1/2} \tag{B17}$$

and

$$\sigma_{\Delta y_v} = \frac{1}{\Delta} \left(\sin^4 \gamma + a_{12}^2 \sin^2 \gamma \right)^{1/2} . \tag{B18}$$

The effective errors in position based on measurement errors were obtained in Appendix A as

$$\sigma_{\Delta x_m} = \frac{1}{\Delta} \left(a_{22}^2 + a_{12}^2 \right)^{1/2} \sigma_m ,$$

and

$$\sigma_{\Delta y_m} = \frac{1}{\Delta} \sigma_m .$$

The ratios $\sigma_{\Delta x_v}/\sigma_{\Delta x_m}$ and $\sigma_{\Delta y_v}/\sigma_{\Delta y_m}$ then express the increase in the error of the position due to the velocity error.

These ratios, expressed as

$$R_1 = \frac{\sigma_{\Delta x_v}}{\sigma_{\Delta x_m}} = \frac{a_{12} \left(\sin^2 \gamma + a_{12}^2 \right)^{1/2}}{\left(a_{22}^2 + a_{12}^2 \right)^{1/2}} \tag{B19}$$

and

$$R_2 = \frac{\sigma_{\Delta y_v}}{\sigma_{\Delta y_m}} = \sin \gamma \ \left(\sin^2 \gamma + a_{12}^2 \right)^{1/2} , \tag{B20}$$

are computed as a function of time and position and presented in Tables B1 through B7. For velocity errors that are large relative to the measurement errors, the numbers of the table are multiplied by $(\sigma_{\Delta v}/\omega_e)/\sigma_m$ to indicate the normalized position error introduced by the velocity error. The computer printout is the program used to solve Eqs. (B19) and (B20).

Computer Program Used to Solve Eqs. (B19) and (B20)

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```
      PROGRAM  COVERAGE
      DIMENSION  SIGX(19,7,19),SIGY(19,7,19),IPRINT(3)
      DIMENSION  R1(19,7,19),R2(19,7,19),R3(19,7,19),R4(19,7,19)
      COMMON/1/SIGX,R1,R2
      COMMON/2/SIGY,R3,R4
      TYPE INTEGER  DCLN,HA
      TYPE DOUBLE   DEC,PHI,GAMMA,TAND,SINP,COSP,COSG,DELTA,COSD,SIND
      TYPE DOUBLE  NUMERATR
      TYPE DOUBLE  SING
      DATA (IPRINT = 8H(***,5X,,0,6HF5.3)))
  900 FORMAT (*1        ERROR IN LONGITUDE       FOR     DEC =*,I4,/)
  901 FORMAT (*0*,5X,*HA(+OR-)  0    15    30    45    60    75    90
     *105   120   135   150   165   180*,/,*   PHI *,/)
  902 FORMAT (***,I5)
  903 FORMAT (*1        ERROR IN LATITUDE        FOR     DEC =*,I4,/)
  904 FORMAT (*0*)
  905 FORMAT (79H1        R1 = A12*((A12**2 + SING**2)/(A22**2 + A12**2))
     ***1/2     FOR     DEC = ,I4,/)
  906 FORMAT (60H1        R2 = SING*(A12**2 + SING**2)**1/2      FOR      D
     *EC = ,I4,/)
  907 FORMAT (*1        EFFECT OF V SUB Z  ON DELTA Y  FOR  DEC =*,I4,/)
  908 FORMAT (*1        EFFECT OF V SUB Y  ON DELTA Y  FOR  DEC =*,I4,/)
  110 FORMAT (*5X,*,I2,*(1X*)
      DTR = 0.01745329251994329 6D
      DO 100   J=1,7
      DCLN = 9*(1-J) + 27
      DEC = DCLN*DTR
      COSD = DCOS(DEC)
      SIND = DSIN(DEC)
      TAND = SIND/COSD
      DO 50   I = 1,19
      LAT = 10*(1 - I) + 90
      PHI = LAT*DTR
      SINP = DSIN(PHI)
      COSP = DCOS(PHI)
      DO 30   K = 1,13
      HA = 15*(K - 1)
      GAMMA = HA*DTR
      COSG = DCOS(GAMMA)
      SING = DSIN(GAMMA)
      A12 = ABSF(COSG*SINP - COSP*TAND)
      A22 = -SINP*SING
      IF(SINP*SIND + COSP*COSD*COSG .GT. 0)  GO TO 20
      SIGX(I,J,K) = SIGY(I,J,K) = R1(I,J,K) = R2(I,J,K) = R3(I,J,K) =
     * R4(I,J,K) = 0.
      GO TO 30
   20 DELTA = SINP - COSP*COSG*TAND
      IF(DELTA .EQ. 0)   DELTA = 1.0D-30
      R1(I,J,K) = A12*SQRTF(A12**2 + SING**2)/SQRTF(A22**2 + A12**2)
      R2(I,J,K) =SING*SQRTF(A12**2 + SING**2)
      SIGX(I,J,K) = SQRTF(1. + R1(I,J,K)**2)
      SIGY(I,J,K) = SQRTF(1. + R2(I,J,K)**2)
   30 CONTINUE
   50 CONTINUE
      PRINT 905,   DCLN
      PRINT 901
```

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```
      DO135   I = 1,19
      LAT = 10*(1 - I) + 90
      PRINT 902, LAT
      M = 0
      DO 133  L=1,13
      K = 14 - L
      IF(R1(I,J,K) .NE. 0,)  GO TO 134
  133 K = K - 1
  134 M = K
      ENCODE (8,110,IPRINT(2)) M
      PRINT IPRINT,       (R1  (I,J,K),K=1,M)
  135 PRINT 904
      PRINT 906,   DCLN
      PRINT 901
      DO235   I = 1,19
      LAT = 10*(1 - I) + 90
      PRINT 902, LAT
      M = 0
      DO 233  L=1,13
      K = 14 - L
      IF(R2(I,J,K) .NE. 0,)  GO TO 234
  233 K = K - 1
  234 M = K
      ENCODE (8,110,IPRINT(2)) M
      PRINT IPRINT,       (R2  (I,J,K),K=1,M)
  235 PRINT 904
      PRINT 900,  DCLN
      PRINT 901
      DO 35   I = 1,19
      LAT = 10*(1 - I) + 90
      PRINT 902, LAT
      M = 0
      DO 33  K=1,13
      IF(SIGX(I,J,K) .EQ. 0.)  GO TO 34
   33 M = M + 1
   34 IF(M .EQ. 0)  GO TO 35
      ENCODE (8,110,IPRINT(2)) M
      PRINT IPRINT,       (SIGX(I,J,K),K=1,M)
   35 PRINT 904
      PRINT 903,  DCLN
      PRINT 901
      DO 40   I = 1,19
      LAT = 10*(1 - I) + 90
      PRINT 902, LAT
      M = 0
      DO 38  K=1,13
      IF(SIGY(I,J,K) .EQ. 0.)  GO TO 39
   38 M = M + 1
   39 IF(M .EQ. 0)  GO TO 40
      ENCODE (8,110,IPRINT(2)) M
      PRINT IPRINT,       (SIGY(I,J,K),K=1,M)
   40 PRINT 904
  100 CONTINUE
      END
```

Table B1a
Position Error in the x Coordinate Caused by the Velocity Error for $\delta = 27°$

R1 = A12*((A12**2 + SING**2)/(A22**2 + A12**2))**1/2 FOR DEC = 27

HA(+OR-) PHI	0	15	30	45	60	75	90	105	120	135	150	165	180
90	1,000	0,966	0,866	0,707	0,500	0,259	0,000	0,259	0,500	0,707	0,866	0,966	1,000
80	0,896	0,864	0,768	0,613	0,409	0,169	0,090	0,348	0,587	0,790	0,944	1,041	1,073
70	0,765	0,738	0,654	0,511	0,312	0,073	0,185	0,440	0,670	0,860	1,000	1,085	1,114
60	0,611	0,594	0,530	0,400	0,204	0,035	0,291	0,536	0,747	0,914	1,031		
50	0,439	0,439	0,397	0,272	0,072	0,168	0,414	0,637	0,819				
40	0,252	0,281	0,242	0,099	0,107	0,336	0,557	0,744					
30	0,059	0,080	0,016	0,172	0,358	0,551	0,723	0,856					
20	0,137	0,256	0,389	0,522	0,662	0,795	0,902						
10	0,328	0,419	0,593	0,763	0,903	1,003	1,057						
0	0,510	0,571	0,714	0,872	1,005	1,092	1,122						
-10	0,675	0,716	0,815	0,926	1,015	1,061							
-20	0,821	0,845	0,901	0,957	0,985	0,968							
-30	0,941	0,951	0,968	0,972	0,939								
-40	1,033	1,030	1,014	0,970	0,883								
-50	1,094	1,080	1,035	0,951									
-60	1,121	1,099											
-70													
-80													
-90													

Table B1b
Position Error in the y Coordinate Caused by the Velocity Error for $\delta = 27°$

R2 = SING*(A12**2 + SING**2)**1/2 FOR DEC = 27

PHI \ HA(+OR-)	0	15	30	45	60	75	90	105	120	135	150	165	180
90	0.000	0.259	0.500	0.707	0.866	0.966	1.000	0.966	0.866	0.707	0.500	0.259	0.000
80	0.000	0.233	0.457	0.659	0.828	0.947	1.004	0.990	0.903	0.747	0.533	0.277	0.000
70	0.000	0.201	0.406	0.608	0.792	0.935	1.015	1.016	0.935	0.776	0.554	0.288	0.000
60	0.000	0.165	0.352	0.560	0.766	0.933	1.032	1.041	0.958	0.791	0.561		
50	0.000	0.126	0.301	0.522	0.752	0.941	1.052	1.062	0.970				
40	0.000	0.090	0.263	0.502	0.752	0.958	1.073	1.077					
30	0.000	0.068	0.250	0.504	0.768	0.980	1.093	1.084					
20	0.000	0.077	0.266	0.527	0.796	1.006	1.109						
10	0.000	0.109	0.306	0.567	0.832	1.032	1.119						
0	0.000	0.148	0.357	0.616	0.870	1.055	1.122						
-10	0.000	0.186	0.411	0.667	0.907	1.072							
-20	0.000	0.220	0.461	0.714	0.938	1.082							
-30	0.000	0.248	0.504	0.752	0.960								
-40	0.000	0.270	0.535	0.779	0.971								
-50	0.000	0.284	0.555	0.792									
-60	0.000	0.290											
-70													
-80													
-90													

Table B2a

Position Error in the x Coordinate Caused by the Velocity Error for $\delta = 18°$

R1 = A12*((A12**2 + SING**2)/(A22**2 + A12**2))**1/2 FOR DEC = 18

PHI \ HA(+OR-)	0	15	30	45	60	75	90	105	120	135	150	165	180
90	1,000	0.966	0.866	0.707	0,500	0.259	0.000	.259	0,500	0.707	0.866	0,966	1.000
80	0,928	0.896	0.800	0.645	0,441	0.201	0.057	.316	0,555	0.758	0.912	1,009	1.041
70	0,829	0.801	0.717	0.575	0,378	0.140	0.118	.374	0,606	0.797	0.937		
60	0,704	0.685	0.621	0.496	0,308	0,071	0.187	.436	0,654				
50	0,557	0.553	0.517	0.409	0,224	0,014	0.269	.505					
40	0,394	0.414	0.406	0.304	0,112	0.128	0.372	.585					
30	0,219	0.276	0.271	0.142	0,063	0,294	0.509						
20	0,037	0.071	0.027	0.180	0,362	0,543	0.696						
10	0,146	0.288	0.470	0.623	0,754	0,857	0.923						
0	0,325	0.415	0.596	0.778	0,925	1.019	1.051						
-10	0,494	0.550	0.675	0.804	0,897	0.938							
-20	0,647	0.680	0.752	0.818	0,839	0.799							
-30	0,781	0.796	0.823	0.830	0,788	0,680							
-40	0,892	0.891	0.881	0.838	0,742								
-50	0,975	0.963	0.921	0.836	0,698								
-60	1,028	1.007	0.940	0.823									
-70	1,051	1.022											
-80													
-90													

Table B2b
Position Error in the y Coordinate Caused by the Velocity Error for $\delta = 18°$

R2 = SING*(A12**2 + SING**2)**1/2 FOR DEC = 18

PHI \ HA(+OR-)	0	15	30	45	60	75	90	105	120	135	150	165	180
90	0.000	0.259	0.500	0.707	0.866	0.966	1.000	0.966	0.866	0.707	0.500	0.259	0.000
80	0.000	0.241	0.470	0.674	0.840	0.953	1.002	0.980	0.888	0.730	0.519	0.269	0.000
70	0.000	0.217	0.431	0.635	0.812	0.942	1.006	0.994	0.903	0.742	0.526		
60	0.000	0.187	0.386	0.593	0.786	0.935	1.013	1.005	0.910				
50	0.000	0.153	0.338	0.553	0.765	0.933	1.022	1.013					
40	0.000	0.117	0.294	0.521	0.753	0.936	1.031	1.016					
30	0.000	0.085	0.261	0.503	0.750	0.944	1.039						
20	0.000	0.067	0.250	0.502	0.759	0.956	1.046						
10	0.000	0.078	0.264	0.519	0.777	0.970	1.050						
0	0.000	0.108	0.298	0.550	0.801	0.984	1.051						
-10	0.000	0.143	0.343	0.590	0.829	0.997							
-20	0.000	0.178	0.391	0.632	0.856	1.008							
-30	0.000	0.209	0.436	0.672	0.880	1.014							
-40	0.000	0.235	0.474	0.705	0.898								
-50	0.000	0.255	0.503	0.729	0.908								
-60	0.000	0.267	0.520	0.742									
-70	0.000	0.272											
-80													
-90													

Table B3a

Position Error in the x Coordinate Caused by the Velocity Error for $\delta = 9°$

R1 = A12*((A12**2 + SING**2)/(A22**2 + A12**2))**1/2 FØR DEC = 9

HA(+ØR-)	0	15	30	45	60	75	90	105	120	135	150	165	180
PHI													
90	1,000	0.966	0.866	0.707	0,500	0.259	0.000	0,259	0.500	0.707	0.866	0.966	1.000
80	0,957	0.925	0.829	0.674	0,470	0.231	0.028	0.286	0.526	0.729	0.884		
70	0,886	0.858	0.773	0.632	0,437	0.201	0,058	0,315					
60	0,787	0.767	0.703	0.582	0,399	0.167	0,091	0,345					
50	0,664	0.658	0,621	0.525	0,355	0.125	0.132						
40	0,521	0.534	0,533	0.462	0,301	0.070	0.187						
30	0,363	0.405	0.444	0.386	0,220	0.016	0.267						
20	0,193	0.284	0.340	0.256	0,065	0.174	0.403						
10	0,018	0.066	0.032	0.185	0,363	0.537	0.676						
0	0.158	0.303	0.524	0.725	0,880	0.979	1.012						
-10	0,330	0.411	0.564	0.696	0,765	0.757							
-20	0,491	0.536	0,625	0.687	0,677	0.580							
-30	0,637	0.658	0.695	0.698	0,632	0.484							
-40	0,764	0.767	0.761	0.716	0,605	0.424							
-50	0,868	0.857	0.817	0.731	0,586	0.380							
-60	0,945	0.924	0.858	0.740	0,567								
-70	0,994	0.966	0.881	0.740	0,548								
-80	1,012	0.980											
-90													

Table B3b
Position Error in the y Coordinate Caused by the Velocity Error for $\delta = 9°$

R2 = SING*(A12**2 + SING**2)**1/2 FOR DEC = 9

HA(+OR-) / PHI	0	15	30	45	60	75	90	105	120	135	150	165	180
90	0.000	0.259	0.500	0.707	0.866	0.966	1.000	0.966	0.866	0.707	0.500	0.259	0.000
80	0.000	0.248	0.483	0.688	0.851	0.959	1.000	0.972	0.875	0.716	0.506		
70	0.000	0.231	0.455	0.660	0.832	0.951	1.001	0.976					
60	0.000	0.207	0.418	0.626	0.810	0.943	1.003	0.978					
50	0.000	0.178	0.376	0.589	0.789	0.938	1.005						
40	0.000	0.146	0.331	0.553	0.770	0.934	1.007						
30	0.000	0.112	0.290	0.523	0.756	0.933	1.009						
20	0.000	0.082	0.261	0.504	0.750	0.935	1.011						
10	0.000	0.067	0.250	0.501	0.752	0.939	1.012						
0	0.000	0.079	0.262	0.512	0.762	0.945	1.012						
-10	0.000	0.107	0.293	0.537	0.779	0.953							
-20	0.000	0.141	0.335	0.571	0.800	0.961							
-30	0.000	0.174	0.379	0.609	0.822	0.968							
-40	0.000	0.203	0.421	0.645	0.842	0.974							
-50	0.000	0.228	0.457	0.676	0.860	0.977							
-60	0.000	0.246	0.484	0.699	0.871								
-70	0.000	0.258	0.501	0.713	0.877								
-80	0.000	0.262											
-90													

Table B4a

Position Error in the x Coordinate Caused by the Velocity Error for $\delta = 0°$

R1 = A12*((A12**2 + SING**2)/(A22**2 + A12**2))**1/2 FØR DEC = 0

PHI \ HA(+ØR-)	0	15	30	45	60	75	90	105	120	135	150	165	180
90	1,000	0.966	0.866	0.707	0,500	0.259	0.000						
80	0,985	0.952	0.856	0.702	0,498	0.259	0.000						
70	0,940	0.912	0.827	0.686	0,493	0.258	0.000						
60	0,866	0.846	0.781	0.661	0,484	0.257	0.000						
50	0,766	0.757	0.719	0.630	0,473	0.255	0.000						
40	0,643	0.650	0.648	0.594	0,462	0.254	0.000						
30	0,500	0.529	0.573	0.559	0,451	0.252	0.000						
20	0,342	0.405	0.503	0.528	0,441	0.251	0.000						
10	0,174	0.298	0.452	0.507	0,435	0.250	0.000						
0													
-10	0,174	0.298	0.452	0.507	0,435	0.250	0.000						
-20	0,342	0.405	0.503	0.528	0,441	0.251	0.000						
-30	0,500	0.529	0.573	0.559	0,451	0.252	0.000						
-40	0,643	0.650	0.648	0.594	0,462	0.254	0.000						
-50	0,766	0.757	0.719	0.630	0,473	0.255	0.000						
-60	0,866	0.846	0.781	0.661	0,484	0.257	0.000						
-70	0,940	0.912	0.827	0.686	0,493	0.258	0.000						
-80	0,985	0.952	0.856	0.702	0,498	0.259	0.000						
-90	1,000	0.966	0.866	0.707	0,500	0.259	0.000						

Table B4b
Position Error in the y Coordinate Caused by the Velocity Error for $\delta = 0°$

R2 = SING*(A12**2 + SING**2)**1/2 FOR DEC = 0

HA(+OR-) PHI	0	15	30	45	60	/5	90	105	120	135	150	165	180
90	0,000	0.259	0.500	0.707	0,866	0,966	1,000						
80	0,000	0.255	0,494	0.702	0,863	0,965	1.000						
70	0,000	0,244	0.478	0.686	0,853	0,962	1.000						
60	0,000	0,227	0,451	0.661	0,839	0,958	1.000						
50	0,000	0.203	0.415	0.630	0,820	0,952	1.000						
40	0,000	0.174	0.374	0.594	0,800	0,947	1.000						
30	0.000	0.142	0.331	0.559	0,781	0.941	1.000						
20	0,000	0.109	0,291	0.528	0,764	0.937	1.000						
10	0,000	0.080	0.261	0.507	0,754	0,934	1,000						
0	0,000	0.067	0,250	0.500	0,750	0,933	1.000						
-10	0,000	0.080	0.261	0.507	0,754	0,934	1,000						
-20	0,000	0.109	0.291	0.528	0,764	0,937	1.000						
-30	0,000	0.142	0.331	0.559	0,781	0,941	1,000						
-40	0,000	0,174	0,374	0.594	0,800	0.947	1,000						
-50	0,000	0.203	0.415	0.630	0,820	0,952	1.000						
-60	0,000	0.227	0,451	0.661	0,839	0,958	1,000						
-70	0,000	0.244	0.478	0.686	0,853	0,962	1,000						
-80	0,000	0,255	0.494	0.702	0,863	0,965	1,000						
-90	0,000	0.259	0.500	0.707	0,866	0,966	1.000						

Table B5a

Position Error in the x Coordinate Caused by the Velocity Error for $\delta = 9^\circ$

R1 = A12*((A12**2 + SING**2)/(A22**2 + A12**2))**1/2   FOR   DEC =   •9

HA(+OR-) PHI	0	15	30	45	60	75	90	105	120	135	150	165	180
90													
80	1,012	0.980											
70	0,994	0.966	0.881	0.740	0,548								
60	0,945	0.924	0.858	0.740	0,567								
50	0,868	0.857	0.817	0.731	0,586	0.380							
40	0,764	0.767	0.761	0.716	0,605	0.424							
30	0,637	0.658	0.695	0.698	0,632	0.484							
20	0,491	0.536	0.625	0.687	0,677	0.580							
10	0,330	0.411	0.564	0.696	0,765	0.757							
0	0,158	0.303	0.524	0.725	0,880	0.979	1.012						
-10	0,018	0.066	0.032	0.185	0,363	0.537	0.676						
-20	0,193	0.284	0.340	0.256	0,065	0.174	0.403						
-30	0,363	0.405	0.444	0.386	0,220	0.016	0.267						
-40	0,521	0.534	0.533	0.462	0,301	0.070	0.187						
-50	0,664	0.658	0.621	0.525	0,355	0.125	0.132						
-60	0,787	0.767	0.703	0.582	0,399	0.167	0.091	0,345					
-70	0,886	0.858	0.773	0.632	0,437	0.201	0.058	0,315					
-80	0,957	0.925	0.829	0.674	0,470	0.231	0,028	0,286	0.526	0.729	0.884		
-90	1.000	0.966	0.866	0.707	0,500	0.259	0,000	0,259	0.500	0.707	0.866	0,966	1.000

Table B5b

Position Error in the v Coordinate Caused by the Velocity Error for $\delta = 9°$

R2 = SING*(A12**2 + SING**2)**1/2 FOR DEC = -9

PHI \ HA(+OR-)	0	15	30	45	60	75	90	105	120	135	150	165	180
90													
80	0.000	0.262											
70	0.000	0.258	0.501	0.713	0.877								
60	0.000	0.246	0.484	0.699	0.871								
50	0.000	0.228	0.457	0.676	0.860	0.977							
40	0.000	0.203	0.421	0.645	0.842	0.974							
30	0.000	0.174	0.379	0.609	0.822	0.968							
20	0.000	0.141	0.335	0.571	0.800	0.961							
10	0.000	0.107	0.293	0.537	0.779	0.953							
0	0.000	0.079	0.262	0.512	0.762	0.945	1.012						
-10	0.000	0.067	0.250	0.501	0.752	0.939	1.012						
-20	0.000	0.082	0.261	0.504	0.750	0.935	1.011						
-30	0.000	0.112	0.290	0.523	0.756	0.933	1.009						
-40	0.000	0.146	0.331	0.553	0.770	0.934	1.007						
-50	0.000	0.178	0.376	0.589	0.789	0.938	1.005						
-60	0.000	0.207	0.418	0.626	0.810	0.943	1.003	0.978					
-70	0.000	0.231	0.455	0.660	0.832	0.951	1.001	0.976					
-80	0.000	0.248	0.483	0.688	0.851	0.959	1.000	0.972	0.875	0.716	0.506		
-90	0.000	0.259	0.500	0.707	0.866	0.966	1.000	0.966	0.866	0.707	0.500	0.259	0.000

Table B6a
Position Error in the x Coordinate Caused by the Velocity Error for $\delta = 18°$

R1 = A12*((A12**2 + SING**2)/(A22**2 + A12**2))**1/2   FØR   DEC = -18

PHI \ HA(+ØR-)	0	15	30	45	60	75	90	105	120	135	150	165	180
90													
80													
70	1.051	1.022											
60	1.028	1.007	0.940	0.823									
50	0.975	0.963	0.921	0.836	0.698								
40	0.892	0.891	0.881	0.838	0.742								
30	0.781	0.796	0.823	0.830	0.788	0.680							
20	0.647	0.680	0.752	0.818	0.839	0.799							
10	0.494	0.550	0.675	0.804	0.897	0.938							
0	0.325	0.415	0.596	0.778	0.925	1.019	1.051						
-10	0.146	0.288	0.470	0.623	0.754	0.857	0.923						
-20	0.037	0.071	0.027	0.180	0.362	0.543	0.696						
-30	0.219	0.276	0.271	0.142	0.063	0.294	0.509						
-40	0.394	0.414	0.406	0.304	0.112	0.128	0.372	0.585					
-50	0.557	0.553	0.517	0.409	0.224	0.014	0.269	0.505					
-60	0.704	0.685	0.621	0.496	0.308	0.071	0.187	0.436	0.654				
-70	0.829	0.801	0.717	0.575	0.378	0.140	0.118	0.374	0.606	0.797	0.937		
-80	0.928	0.896	0.800	0.645	0.441	0.201	0.057	0.316	0.555	0.758	0.912	1.009	1.041
-90	1.000	0.966	0.866	0.707	0.500	0.259	0.000	0.259	0.500	0.707	0.866	0.966	1.000

Table B6b

Position Error in the y Coordinate Caused by the Velocity Error for $\delta = 18°$

R2 = SING*(A12**2 + SING**2)**1/2   FOR   DEC = -18

PHI \ HA(+OR-)	0	15	30	45	60	75	90	105	120	135	150	165	180
90													
80													
70	0,000	0.272											
60	0,000	0.267	0.520	0.742									
50	0,000	0.255	0.503	0.729	0,908								
40	0,000	0.235	0.474	0.705	0,898								
30	0,000	0.209	0.436	0.672	0,880	1.014							
20	0,000	0.178	0.391	0.632	0,856	1,008							
10	0,000	0.143	0.343	0.590	0,829	0.997							
0	0,000	0.108	0.298	0.550	0,801	0.984	1.051						
-10	0,000	0.078	0.264	0.519	0.777	0.970	1.050						
-20	0,000	0,067	0.250	0.502	0.759	0,956	1.046						
-30	0,000	0,085	0.261	0.503	0.750	0.944	1.039						
-40	0,000	0.117	0.294	0.521	0.753	0.936	1,031	1,016					
-50	0,000	0.153	0.338	0.553	0.765	0,933	1,022	1,013					
-60	0,000	0.187	0.386	0.593	0.786	0.935	1.013	1,005	0,910				
-70	0,000	0.217	0.431	0,635	0,812	0.942	1.006	0,994	0,903	0,742	0,526		
-80	0,000	0.241	0.470	0,674	0,840	0,953	1,002	0,980	0,888	0,730	0,519	0,269	0,000
-90	0,000	0.259	0,500	0.707	0,866	0,966	1,000	0,966	0,866	0.707	0,500	0.259	0.000

Table B7a
Position Error in the x Coordinate Caused by the Velocity Error for $\delta = 27°$

R1 = A12*((A12**2 + SING**2)/(A22**2 + A12**2))**1/2 FØR DEC = -27

HA(+ØR-) PHI	0	15	30	45	60	75	90	105	120	135	150	165	180
90													
80													
70													
60	1,121	1.099											
50	1,094	1.080	1.035	0.951									
40	1,033	1.030	1.014	0.970	0,883								
30	0,941	0.951	0.968	0.972	0,935								
20	0,821	0.845	0.901	0.957	0,985	0.968							
10	0,675	0.716	0.815	0.926	1,015	1.061							
0	0,510	0.571	0.714	0.872	1,005	1.092	1.122						
-10	0,328	0.419	0.593	0.763	0,903	1.003	1.057						
-20	0,137	0.256	0.389	0.522	0,662	0.795	0.902						
-30	0,059	0.080	0.016	0.172	0,358	0.551	0.723	0,856					
-40	0,252	0.281	0.242	0.099	0,107	0.336	0.557	0,744					
-50	0,439	0.439	0.397	0.272	0,072	0.168	0.414	0,637	0,819				
-60	0,611	0.594	0.530	0.400	0,204	0.035	0.291	0,536	0,747	0,914	1.031		
-70	0,765	0.738	0.654	0.511	0,312	0.073	0.185	0,440	0,670	0,860	1.000	1,085	1.114
-80	0,896	0.864	0.768	0.613	0,409	0.169	0.090	0,348	0,587	0,790	0.944	1,041	1.073
-90	1,000	0.966	0.866	0.707	0,500	0.259	0.000	0,259	0,500	0,707	0.866	0,966	1.000

Table B7b

Position Error in the y Coordinate Caused by the Velocity Error for $\delta = 27°$

R2 = SING*(A12**2 + SING**2)**1/2 FOR DEC = -27

HA(+OR-) PHI	0	15	30	45	60	75	90	105	120	135	150	165	180
90													
80													
70													
60	0,000	0.290											
50	0,000	0.284	0.555	0.792									
40	0,000	0.270	0.535	0.779	0,971								
30	0,000	0.248	0.504	0.752	0,960								
20	0,000	0.220	0.461	0.714	0,938	1.082							
10	0,000	0.186	0.411	0.667	0,907	1,072							
0	0,000	0.148	0.357	0.616	0,870	1.055	1.122						
-10	0,000	0.109	0.306	0.567	0,832	1.032	1,119						
-20	0,000	0.077	0.266	0.527	0,796	1.006	1.109						
-30	0,000	0.068	0.250	0.504	0,768	0.980	1.093	1,084					
-40	0,000	0.090	0.263	0.502	0,752	0,958	1.073	1,077					
-50	0,000	0.126	0.301	0.522	0,752	0,941	1,052	1,062	0,970				
-60	0,000	0.165	0.352	0.560	0,766	0,933	1,032	1,041	0,958	0.791	0.561		
-70	0,000	0.201	0.406	0.608	0,792	0,935	1.015	1,016	0,935	0.776	0.554	0,288	0.000
-80	0,000	0.233	0.457	0.659	0,828	0,947	1.004	0,990	0,903	0.747	0.533	0,277	0.000
-90	0,000	0.259	0,500	0.707	0,866	0,966	1.000	0,966	0,866	0.707	0.500	0,259	0.000

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* Naval Research Laboratory Washington, D.C. 20390	2a. REPORT SECURITY CLASSIFICATION 2b. GROUP 3

3. REPORT TITLE [Unclassified Title]

A LUNAR RADAR NAVIGATION CONCEPT

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*
A final report on one phase of the problem; work is continuing.

5. AUTHOR(S) *(First name, middle initial, last name)*

A. Shapiro, E.A. Uliana, and B.S. Yaplee

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
	60	4

8a. CONTRACT OR GRANT NO. NRL Problem A01-35 b. PROJECT NO. 33404, Task 4896 c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) NRL Report 6814 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*

10. DISTRIBUTION STATEMENT
In addition to security requirements which apply to this document and must be met, each transmittal outside the agencies of the U.S. Government must have prior approval of the Director, Naval Research Laboratory, Washington, D.C. 20390.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY

13. ABSTRACT

A navigation system concept is described that utilizes the moon as a reflector in a bistatic radar system. By measuring the range and range rate of the receiver relative to the moon, the location of the receiver on the earth's surface can be determined in both longitude and latitude. Lunar radar observations have shown that lunar range and range rate measurement accuracies equivalent to ±30 m can be achieved. By placing a transponder on the moon, the basic measurement accuracy could be improved by a factor of 5. However, for operational measurements where rapid readout is required, these accuracies would probably be degraded by a factor of 5 to 10. Placing three transmitters at appropriate locations on the earth's surface will provide worldwide coverage. A transmitter with an average power of 2 MW and a transmitting aperture of 170 m would supply a S/N ratio sufficient for reliable position determination with a dipole antenna receiver. One possible radar receiver configuration incorporating both a search and track mode is given.

A mathematical analysis of the coverage and the effective position accuracy indicates that (a) worldwide coverage is available, (b) time coverage is restricted to 50 percent on the average, but the time distribution of the coverage varies over a monthly period, (c) at low latitudes the effective location accuracy is a function of the moon's declination, and (d) the optimum accuracy is obtained at high latitudes.

(over)

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Bistatic radar						
Lunar navigation concept						
Lunar radar						

While restricted to some extent in coverage as described above, and more sensitive to systematic errors in the ship's velocity, the lunar radar navigation system, when compared with other worldwide radio navigation systems such as Omega and Transit, can achieve higher accuracies, is less vulnerable to jamming, and can also provide an independent, one-way communication channel. With improved technological developments and active reflectors on the moon, position accuracies of the order of several meters appear possible.